Appendix for "Duopolistic Competition and Monetary Policy"

Kozo Ueda

August 10, 2022

A Model

A.1 Demand System

We consider an arbitrary invertible demand system $x_t^i = x^i(p_t^i, p_t^{-i}; M_t) = x^i(p_t^i/M_t, p_t^{-i}/M_t)$ for firm i. Firm profit is given by $\Pi_t^i = (p_t^i - W_t)x^i(p_t^i/M_t, p_t^{-i}/M_t)$. We assume that nominal spending must be equal to the money supply, which yields $P_tC_t = M_t = W_t$. Given symmetry, demand is shared equally between firms, which equals $x^i \equiv M_t/(nP_t) = 1/(np)$ in steady state. We define

$$\Psi^i \equiv \frac{\partial \log x^i(\bar{p}^i/M, p^{-i}/M)}{\partial \log(\bar{p}^i/M)} \tag{1}$$

$$\Psi^{-i} \equiv \frac{\partial \log x^i(\bar{p}^i/M, p^{-i}/M)}{\partial \log(p^{-i}/M)} \tag{2}$$

$$\Psi^{i,i} \equiv \frac{\partial \Psi^i}{\partial \log(\bar{p}^i/M)} \tag{3}$$

$$\Psi^{-i,-i} \equiv \frac{\partial \Psi^{-i}}{\partial \log(\bar{p}^{-i}/M)} \tag{4}$$

$$\Psi^{i,-i} \equiv \frac{\partial \Psi^{-i}}{\partial \log(\bar{p}^i/M)} = \frac{\partial \Psi^i}{\partial \log(\bar{p}^{-i}/M)}. \tag{5}$$

CES and Monopolistic Competition We assume that the number of firms n is infinite instead of two. For each product line j, consumption is aggregated based on the CES form: $c_t^j = \left\{\sum_{i=1}^n (1/n)^{1/\sigma} \left(x_t^i\right)^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}$. This yields demand $x_t^i = \frac{1}{n}\left(\frac{p_t^i}{P_t}\right)^{-\sigma}\frac{M_t}{P_t}$ and $P_t = \left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{\frac{1}{1-\sigma}}$. Thus, $\log x_t^i(p_t^i/M_t) = c - \sigma \log(p_t^i/M_t)$, which leads to

$$\Psi^i = -\sigma,$$

$$\Psi^{-i} = \Psi^{i,i} = \Psi^{-i,-i} = \Psi^{i,-i} = 0.$$

CES and Oligopolistic Competition In the case of CES preferences, for each product line j, consumption is aggregated following the CES form of aggregation: $c_t^j = \left\{\sum_{i=1}^n (1/n)^{1/\sigma}\left(x_t^i\right)^{\frac{\sigma-1}{\sigma}}\right\}^{\frac{\sigma}{\sigma-1}}$. This yields demand $x_t^i = \frac{1}{n}\left(\frac{p_t^i}{P_t}\right)^{-\sigma}\frac{M_t}{P_t}$ and $P_t = \left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{\frac{1}{1-\sigma}}$. Thus, $\log x_t^i = \log(1/n) - \sigma(\log(p_t^i/M_t) - \log(P_t/M_t)) - \log(P_t/M_t) = \log(1/n) - \sigma\log(p_t^i/M_t) - \log\left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}$,

$$\begin{aligned}
\Psi^i &= -\sigma - \left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)\left(p_t^i/M_t\right)^{-\sigma}\left(p_t^i/M_t\right) \\
&= -\sigma - \left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)\left(p_t^i/M_t\right)^{1-\sigma} \\
&= -\sigma - \frac{1}{n}(1-\sigma)\left\{p^{1-\sigma}\right\}^{-1}p^{1-\sigma} \\
&= -\frac{(n-1)\sigma+1}{n},
\end{aligned}$$

$$\begin{aligned}
\Psi^{-i} &= -\left\{\sum_{i=1}^n \frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)\left(p_t^{-i}/M_t\right)^{1-\sigma} \\
&= \frac{1}{n}(\sigma-1),
\end{aligned}$$

$$
\begin{aligned}
\Psi^{i,i} &= \left\{\sum_{i=1}^{n}\frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-2}\left\{\frac{1}{n}(1-\sigma)\left(p_t^i/M_t\right)^{1-\sigma}\right\}^2 \\
&\quad -\left\{\sum_{i=1}^{n}\frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)(1-\sigma)\left(p_t^i/M_t\right)^{1-\sigma} \\
&= \left\{p^{1-\sigma}\right\}^{-2}\left\{\frac{1}{n}(1-\sigma)\left(p\right)^{1-\sigma}\right\}^2 - \left\{p^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)^2\left(p\right)^{1-\sigma} \\
&= \left\{\frac{1}{n}(1-\sigma)\right\}^2 - \frac{1}{n}(1-\sigma)^2 \\
&= -(n-1)\left\{\frac{1}{n}(1-\sigma)\right\}^2.
\end{aligned}
$$

$$
\begin{aligned}
\Psi^{-i,-i} &= \left\{\sum_{i=1}^{n}\frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-2}\left\{\frac{1}{n}(1-\sigma)\left(p_t^{-i}/M_t\right)^{1-\sigma}\right\}^2 \\
&\quad -\left\{\sum_{i=1}^{n}\frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)(1-\sigma)\left(p_t^{-i}/M_t\right)^{1-\sigma} \\
&= \left\{p^{1-\sigma}\right\}^{-2}\left\{\frac{1}{n}(1-\sigma)\left(p\right)^{1-\sigma}\right\}^2 - \left\{p^{1-\sigma}\right\}^{-1}\frac{1}{n}(1-\sigma)^2\left(p\right)^{1-\sigma} \\
&= -(n-1)\left\{\frac{1}{n}(1-\sigma)\right\}^2.
\end{aligned}
$$

$$
\begin{aligned}
\Psi^{i,-i} &= \left\{\sum_{i=1}^{n}\frac{1}{n}\left(p_t^i\right)^{1-\sigma}\right\}^{-2}\frac{1}{n}(1-\sigma)\left(p_t^{-i}/M_t\right)^{1-\sigma}\frac{1}{n}(1-\sigma)\left(p_t^i/M_t\right)^{1-\sigma} \\
&= \left\{p^{1-\sigma}\right\}^{-2}\left\{\frac{1}{n}(1-\sigma)\left(p\right)^{1-\sigma}\right\}^2 \\
&= \left\{\frac{1}{n}(1-\sigma)\right\}^2,
\end{aligned}
$$

where we use $d(x)/dlogx = d(x)/(dx/x) = x$.

Hotelling Address Model and Duopolistic Competition In the case of the Hotelling model, demand is given by $x_t^i = \left(\frac{1}{2} - \frac{\log(p_t^i/M_t)-\log(p_t^{-i}/M_t)}{2\tau}\right)\frac{M_t}{p_t^i}$. Thus, $\log x_t^i = \log\left(\frac{1}{2} - \frac{\log(p_t^i/M_t)-\log(p_t^{-i}/M_t)}{2\tau}\right) - \log(p_t^i/M_t)$,

$$
\begin{aligned}
\Psi^{i} &= \left(\frac{1}{2} - \frac{\log(p_t^i/M_t)-\log(p_t^{-i}/M_t)}{2\tau}\right)^{-1}\left(-\frac{1}{2\tau}\right) - 1 \\
&= -\frac{1}{\tau} - 1 = -\frac{1+\tau}{\tau},
\end{aligned}
$$

$$
\begin{aligned}
\Psi^{-i} &= \left(\frac{1}{2} - \frac{\log(p_t^i/M_t)-\log(p_t^{-i}/M_t)}{2\tau}\right)^{-1}\left(\frac{1}{2\tau}\right) \\
&= \frac{1}{\tau},
\end{aligned}
$$

$$
\begin{aligned}
\Psi^{i,i} &= -\left(\frac{1}{2} - \frac{\log(p_t^i/M_t)-\log(p_t^{-i}/M_t)}{2\tau}\right)^{-2}\left(-\frac{1}{2\tau}\right)\left(-\frac{1}{2\tau}\right) \\
&= -\frac{1}{\tau^2},
\end{aligned}
$$

Table 1: Comparison of CES, Kinked Demand, and Hotelling Models

	General	CES preferences (monopolistic, $n \to \infty$)	CES preferences (finite n)	Kinked demand	Hotelling ($n = 2$)
Own elasticity Ψ^i		$-\sigma < 0$	$-\frac{(n-1)\sigma+1}{n} < 0$	–	$-\frac{1+\tau}{\tau} < 0$
Cross elasticity Ψ^{-i}		0	$\frac{\sigma-1}{n} > 0$ if $\sigma > 1$	–	$\frac{1}{\tau} > 0$
Own superelasticity $\Psi^{i,i}$		0	$-(n-1)\left(\frac{\sigma-1}{n}\right)^2 < 0$	Lower	$-\frac{1}{\tau^2} < 0$
Cross superelasticity $\Psi^{i,-i}$		0	$\left(\frac{\sigma-1}{n}\right)^2 > 0$	–	$\frac{1}{\tau^2} > 0$
Steady-state markup without price stickiness $p-1$	$\frac{\Psi^i}{\Psi^i+1}$	$\frac{1}{\sigma-1}$	$\frac{n}{(n-1)(\sigma-1)}$	–	τ
Slope of the best response price	$\frac{\Psi^{i,-i}}{\Psi^i(1+\Psi^i)}$	0	$\frac{\sigma-1}{\{(n-1)\sigma+1\}(n-1)}$		$\frac{1}{1+\tau}$

$$\begin{aligned}\Psi^{-i,-i} &= -\left(\frac{1}{2} - \frac{\log(p_t^i/M_t) - \log(p_t^{-i}/M_t)}{2\tau}\right)^{-2}\left(\frac{1}{2\tau}\right)\left(\frac{1}{2\tau}\right)\\ &= -\frac{1}{\tau^2},\end{aligned}$$

$$\begin{aligned}\Psi^{i,-i} &= -\left(\frac{1}{2} - \frac{\log(p_t^i/M_t) - \log(p_t^{-i}/M_t)}{2\tau}\right)^{-2}\left(-\frac{1}{2\tau}\right)\left(\frac{1}{2\tau}\right)\\ &= \frac{1}{\tau^2},\end{aligned}$$

Suppose that Ψ^i is the same for the CES and Hotelling models, that is, $-(\sigma+1)/2 = -(1+\tau)/\tau$, which yields $\sigma - 1 = 2/\tau$. Then, it is clear that the other Ψ's become the identical.

Table 1 provides the comparison of demand elasticities for a general model, CES preferences based on monopolistic competition, CES preferences based on oligopolistic competition, kinked demand, and the Hotelling address model.

A.2 Steady State without Price Stickiness

Before we introduce price stickiness, we consider the equilibrium in a steady state. The first-order condition with respect to p_t^i yields

$$\begin{aligned}\frac{\partial \Pi_t^i}{\partial p_t^i} &= \frac{\partial}{\partial p_t^i}\Big((p_t^i - W_t)x^i(p_t^i/M_t, p_t^{-i}/M_t)\Big)\\ &= x^i(p_t^i/M_t, p_t^{-i}/M_t) + \frac{p_t^i - W_t}{M_t}\frac{\partial x^i}{\partial p_t^i} = 0.\end{aligned}$$

In the steady state with $W = M = 1$, it becomes

$$\begin{aligned}0 &= x^i + (p^i - 1)\frac{x^i \partial \log x^i(p^i, p^{-i})}{p^i \partial \log p^i}\\ 0 &= 1 + (p^i - 1)\frac{\partial \log x^i}{p^i \partial \log p^i}\\ &= 1 + (p^i - 1)\frac{\Psi^i}{p^i}.\end{aligned}$$

This leads to

$$p = \frac{\Psi^i}{\Psi^i + 1}. \tag{6}$$

By differentiating with respect to $\log(p^{-i})$, we obtain the best response of $\log(p^i)$ to $\log(p^{-i})$ as

$$0 = \frac{\partial p^i}{\partial \log(p^{-i})} + \frac{\partial p^i}{\partial \log(p^{-i})}\Psi^i + \left(p^i - 1\right)\frac{\partial \Psi^i}{\partial \log p^{-i}}.$$

$$\frac{\partial \log p^i}{\partial \log p^{-i}} = \frac{\Psi^{i,-i}}{\Psi^i(1+\Psi^i)}. \tag{7}$$

In the case of the Hotelling model, it equals

$$\frac{\partial \log p^i}{\partial \log p^{-i}} = \frac{\frac{1}{\tau^2}}{-\frac{1+\tau}{\tau}(1-\frac{1+\tau}{\tau})} = \frac{1}{1+\tau}. \tag{8}$$

Thus, the degree of strategic complementarity is positive.

A.3 Pricing under Calvo-type Price Stickiness

When firm i has a chance to set its price at t, it sets $\bar{p}_t^i$ to maximize

$$\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}_t^i - W_{t+k}\right)\theta^{k+1} x^i(\bar{p}_t^i/M_{t+k}, p_{t-1}^{-i}/M_{t+k})\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}_t^i - W_{t+k}\right)\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'} x^i(\bar{p}_t^i/M_{t+k}, p_{t+k'}^{-i}/M_{t+k})\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}. \tag{9}$$

The first-order condition for the optimal $\bar{p}_t^i$ is given by

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\theta^{k+1} x^i(\bar{p}_t^i/M_{t+k}, p_{t-1}^{-i}/M_{t+k})\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'} x^i(\bar{p}_t^i/M_{t+k}, p_{t+k'}^{-i}/M_{t+k})\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\bar{p}_t^i - M_{t+k}\right)\left[\theta^{k+1}\frac{\partial x^i(\bar{p}_t^i/M_{t+k}, p_{t-1}^{-i}/M_{t+k})}{\partial \bar{p}_t^i}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\bar{p}_t^i - M_{t+k}\right)\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\frac{\partial x^i(\bar{p}_t^i/M_{t+k}, p_{t+k'}^{-i}/M_{t+k})}{\partial \bar{p}_t^i}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\bar{p}_t^i - M_{t+k}\right)\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\frac{\partial x^i(\bar{p}_t^i/M_{t+k}, p_{t+k'}^{-i}/M_{t+k})}{\partial p_{t+k'}^{-i}}\frac{\partial p_{t+k'}^{-i}}{\partial \bar{p}_t^i}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}.
\end{aligned}$$

Given the Markov perfect equilibrium, the log-linearized optimal reset prices are expressed in the following forms:

$$p_t^{i*} = \Gamma \hat{p}_{t-1}^i + \Gamma^* \hat{p}_{t-1}^{-i} + \Gamma^\varepsilon \varepsilon_t, \tag{10}$$
$$\partial \log \bar{p}_{t+k}^{-i}/\partial \log \bar{p}_t^i = \Gamma^* \text{ for } k \geq 1, \tag{11}$$

where $\bar{p}_t^i \equiv pM_t e^{p_t^{i*}}$, $p_t^i \equiv pM_t e^{\hat{p}_t^i}$. It should be noted that, in general, the partial derivative $\partial \log p^B(p_{t+k-1}^A, p_{t+k-1}^B, x_{t+k})/\partial \log p_t^A$ for $k = 1, 2, \cdots$ depends on the state, not constant at Γ^*. And, $\log p_t^B$ is approximated around steady state ($\log p = \varepsilon = 0$) as

$$\begin{aligned}
\log p_t^B =& \Gamma \log p_{t-1}^B + \Gamma^* \log p_{t-1}^A + \Gamma^\varepsilon \varepsilon_t \\
&+ g_{11}/2\left(\log p_{t-1}^B\right)^2 + g_{22}/2\left(\log p_{t-1}^A\right)^2 + g_{33}\left(\varepsilon_t\right)^2 + g_{12}\left(\log p_{t-1}^B \log p_{t-1}^A\right) + g_{13}\left(\log p_{t-1}^B \varepsilon_t\right) + g_{23}\left(\log p_{t-1}^A \varepsilon_t\right) \\
&+ g_{111}\left(\log p_{t-1}^B\right)^3 + \cdots,
\end{aligned} \tag{12}$$

where ε_t is an aggregate monetary policy shock. This suggests that $\partial \text{log} p^B(p_t^A, p_t^B, \varepsilon_{t+1})/\partial \text{log} p_t^A = \Gamma^* + g_{22}\text{log}p_t^A + g_{12}\text{log}p_t^B + g_{23}\varepsilon_{t+1} + \cdots$. Thus, it generally depends on state (p_t^A, p_t^B, ε_{t+1}). In our study, we make the log-linearization approximation, that is,

$$\text{log}p_t^B = \Gamma \text{log}p_{t-1}^B + \Gamma^* \text{log}p_{t-1}^A + \Gamma^\varepsilon \varepsilon_t, \tag{13}$$

which approximates $\partial \text{log} p^B(p_{t+k-1}^A, p_{t+k-1}^B, x_{t+k})/\partial \text{log} p_t^A = \Gamma^*$ for $k = 1, 2, \cdots$.

Proposition 1 *The method of undetermined coefficients enables us to solve p, Γ, Γ^*, and Γ^ε from the coefficients of 1, $\hat{p}_{t-1}^i$, $\hat{p}_{t-1}^{-i}$, and ε_t in the following equation:*

$$\begin{aligned}
0 =& \frac{1}{1-\theta\beta} \\
&- \Psi^i \frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t + \Psi^i \frac{1}{1-\theta\beta}\left(\Gamma\hat{p}_{t-1}^i + \Gamma^*\hat{p}_{t-1}^{-i} + \Gamma^\varepsilon\varepsilon_t\right) \\
&- \Psi^{-i}\frac{1}{1-\rho}\theta\left(\frac{1}{1-\theta^2\beta} - \frac{\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t + \Psi^{-i}\frac{\theta}{1-\theta^2\beta}\hat{p}_{t-1}^{-i} \\
&+ \Psi^{-i}\frac{1}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ \Psi^{-i}\left(\mathbb{\Gamma}_{k0}^i\hat{p}_{t-1}^i + \mathbb{\Gamma}_{k0}^{-i}\hat{p}_{t-1}^{-i} + \mathbb{\Gamma}_{k0}^\varepsilon\varepsilon_t\right) \\
&+ (1-\frac{1}{p})\Psi^i\frac{1}{1-\theta\beta} \\
&- \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t \\
&+ \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{1}{1-\theta\beta}\left(\Gamma\hat{p}_{t-1}^i + \Gamma^*\hat{p}_{t-1}^{-i} + \Gamma^\varepsilon\varepsilon_t\right) \\
&- (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{1}{1-\rho}\theta\left(\frac{1}{1-\theta^2\beta} - \frac{\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{\theta}{1-\theta^2\beta}\hat{p}_{t-1}^{-i} \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{1}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\mathbb{\Gamma}_{k0}^i\hat{p}_{t-1}^i + \mathbb{\Gamma}_{k0}^{-i}\hat{p}_{t-1}^{-i} + \mathbb{\Gamma}_{k0}^\varepsilon\varepsilon_t) \\
&+ (1-\frac{1}{p})\left(\frac{-1}{1-\theta^2\beta} + \frac{1}{1-\theta\beta}\right)\Psi^{-i}\Gamma^* \\
&- \left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\Gamma^*\left\{\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t - \frac{\rho}{1-\rho}\left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t\right\} \\
&+ \left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^2\beta}{1-\theta^2\beta}\right)\Gamma^*\left(\Gamma\hat{p}_{t-1}^i + \Gamma^*\hat{p}_{t-1}^{-i} + \Gamma^\varepsilon\varepsilon_t\right) \\
&+ (1-\frac{1}{p})\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})\frac{\theta\beta\rho}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ (1-\frac{1}{p})\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\mathbb{\Gamma}_{k1}^i\hat{p}_{t-1}^i + \mathbb{\Gamma}_{k1}^{-i}\hat{p}_{t-1}^{-i} + \mathbb{\Gamma}_{k1}^\varepsilon\varepsilon_t),
\end{aligned} \tag{14}$$

where $\mathbb{\Gamma}_{k0}^i, \mathbb{\Gamma}_{k0}^{-i}, \mathbb{\Gamma}_{k0}^\varepsilon$ are the second row and first to third columns, respectively, of the following matrix:

$$(1-\theta)\Gamma\left[I - (\mathbb{\Gamma}/\theta)\right]^{-1}\left[\frac{1}{1-\theta^2\beta}I - \mathbb{\Gamma}/\theta\left[I - (\theta\beta\mathbb{\Gamma})\right]^{-1}\right],$$

$$\textit{where } \mathbb{\Gamma} = \begin{pmatrix} \Gamma & \Gamma^* & \Gamma^\varepsilon \\ \Gamma^* & \Gamma & \Gamma^\varepsilon \\ 0 & 0 & \rho \end{pmatrix},$$

and $\mathbb{\Gamma}^{i}_{k1}, \mathbb{\Gamma}^{-i}_{k1}, \mathbb{\Gamma}^{\varepsilon}_{k1}$ *are the second row and first to third columns, respectively, of the following matrix:*

$$(1-\theta)\Gamma\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\mathbb{\Gamma}/\theta\left[\frac{\theta^2\beta}{1-\theta^2\beta}I-\theta\beta\mathbb{\Gamma}\left[I-(\theta\beta\mathbb{\Gamma})\right]^{-1}\right].$$

(Proof) Equation (14) can be derived as follows. Note that the term $\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$ equals one because $P_tC_t = M_t$. Thus,

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{M_t}{M_{t+k}}\right)\left[\theta^{k+1}x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})\right] \\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{M_t}{M_{t+k}}\right)\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t+k'}/M_{t+k})\right] \\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{\bar{p}^i_t}{M_{t+k}}-1\right)\frac{M_t}{M_{t+k}}\left[\theta^{k+1}\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{\partial(\bar{p}^i_t/M_{t+k})}\right] \\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{\bar{p}^i_t}{M_{t+k}}-1\right)\frac{M_t}{M_{t+k}}\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t+k'}/M_{t+k})}{\partial(\bar{p}^i_t/M_{t+k})}\right] \\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{\bar{p}^i_t}{M_{t+k}}-1\right)\frac{M_t}{M_{t+k}}\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t+k'}/M_{t+k})}{\partial(p^{-i}_{t+k'}/M_{t+k})}\frac{\partial p^{-i}_{t+k'}}{\partial\bar{p}^i_t}\right].
\end{aligned}$$

In log-linearization, each term in the above equation is given by

$$\begin{aligned}
x^i(\bar{p}^i_t/M_{t+k}, p_t^{-i}/M_{t+k}) =& x^i \\
&+\frac{\partial \log x^i(\bar{p}^i/M, p^{-i}/M)}{\partial\log(\bar{p}^i/M)}x^i d\log(\bar{p}^i_t/M_{t+k}) \\
&+\frac{\partial \log x^i(\bar{p}^i/M, p^{-i}/M)}{\partial\log(p^{-i}/M)}x^i d\log(p_t^{-i}/M_{t+k}) \\
=& x^i\left\{1+\Psi^i(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{-i}(\log(M_t/M_{t+k})+\hat{p}_t^{-i})\right\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{\partial(\bar{p}^i_t/M_{t+k})} =& \frac{x^i_{t+k}}{\bar{p}^i_t/M_{t+k}}\frac{\partial\log x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{\partial\log(\bar{p}^i_t/M_{t+k})} \\
=& x^i\left\{1+\Psi^i(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{-i}(\log(M_{t-1}/M_{t+k})+\hat{p}^{-i}_{t-1})\right\} \\
&\cdot\frac{\partial\log x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{(\bar{p}^i_t/M_{t+k})\partial\log(\bar{p}^i_t/M_{t+k})} \\
=& x^i\left\{1+\Psi^i(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{-i}(\log(M_{t-1}/M_{t+k})+\hat{p}^{-i}_{t-1})\right\} \\
&\cdot\frac{M_{t+k}}{pM_te^{p^{i*}_t}}\cdot\left\{\Psi^i+\Psi^{i,i}(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{i,-i}(\log(M_{t-1}/M_{t+k})+\hat{p}^{-i}_{t-1})\right\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t+k'}/M_{t+k})}{\partial(p^{-i}_{t+k'}/M_{t+k})}\frac{\partial p^{-i}_{t+k'}}{\partial\bar{p}^i_t} =& \frac{x^i_{t+k}}{\bar{p}^{-i}_{t+k'}/M_{t+k}}\frac{\partial\log x^i(\bar{p}^i_t/M_{t+k}, \bar{p}^{-i}_{t+k'}/M_{t+k})}{\partial\log(\bar{p}^{-i}_{t+k'}/M_{t+k})}\frac{p^{-i}_{t+k'}}{\bar{p}^i_t}\frac{\partial\log p^{-i}_{t+k'}}{\partial\log\bar{p}^i_t} \\
=& x^i\left\{1+\Psi^i(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{-i}(\log(M_{t+k'}/M_{t+k})+p^{*-i}_{t+k'})\right\} \\
&\cdot\left\{\Psi^{-i}+\Psi^{i,-i}(\log(M_t/M_{t+k})+p^{i*}_t)+\Psi^{-i,-i}(\log(M_{t+k'}/M_{t+k})+p^{*-i}_{t+k'})\right\} \\
&\cdot\frac{M_{t+k}}{pM_te^{p^{i*}_t}}\cdot\Gamma^*.
\end{aligned}$$

Thus, the first-order condition becomes

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left\{ 1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i} (\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} \left\{ 1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i} (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left(\frac{p M_t e^{p_t^{i*}}}{M_{t+k}} - 1 \right) \\
&\cdot \left\{ 1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i} (\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \right\} \\
&\cdot \frac{M_{t+k}}{p M_t e^{p_t^{i*}}} \cdot \left\{ \Psi^i + \Psi^{i,i} (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{i,-i} (\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{p M_t e^{p_t^{i*}}}{M_{t+k}} - 1 \right) \\
&\cdot \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} \left\{ 1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i} (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right. \\
&\left. \cdot \frac{M_{t+k}}{p M_t e^{p_t^{i*}}} \cdot \left\{ \Psi^i + \Psi^{i,i} (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{i,-i} (\log(M_{t+k'}/M_{t+k}) + \hat{p}_{t+k'}^{-i}) \right\} \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{p M_t e^{p_t^{i*}}}{M_{t+k}} - 1 \right) \\
&\cdot \left[\sum_{k'=1}^{k} (1-\theta) \theta^{k-k'} \left\{ 1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i} (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right. \\
&\left. \cdot \left\{ \Psi^{-i} + \Psi^{i,-i} (\log(M_t/M_{t+k}) + p_t^{i*}) + \Psi^{-i,-i} (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right] \\
&\cdot \frac{M_{t+k}}{p M_t e^{p_t^{i*}}} \cdot \Gamma^*.
\end{aligned}
$$

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[1 + \Psi^i (\log(M_t/M_{t+k}) + p_t^{i*}) \right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left[\Psi^{-i} (\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} \left\{ \Psi^{-i} (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left\{ 1 - \frac{1}{p} + \frac{1}{p} (\log(M_t/M_{t+k}) + p_t^{i*}) \right\} \\
&\cdot \left\{ \Psi^i + (\Psi^i \Psi^i + \Psi^{i,i}) (\log(M_t/M_{t+k}) + p_t^{i*}) + (\Psi^i \Psi^{-i} + \Psi^{i,-i}) (\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left\{ 1 - \frac{1}{p} + \frac{1}{p} (\log(M_t/M_{t+k}) + p_t^{i*}) \right\} \\
&\cdot \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} \left\{ \Psi^i + (\Psi^i \Psi^i + \Psi^{i,i}) (\log(M_t/M_{t+k}) + p_t^{i*}) + (\Psi^i \Psi^{-i} + \Psi^{i,-i}) (\log(M_{t+k'}/M_{t+k}) + \hat{p}_{t+k'}^{-i}) \right\} \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left\{ 1 - \frac{1}{p} + \frac{1}{p} (\log(M_t/M_{t+k}) + p_t^{i*}) \right\} \Gamma^* \\
&\cdot \left[\sum_{k'=1}^{k} (1-\theta) \theta^{k-k'} \left\{ \Psi^{-i} + (\Psi^i \Psi^{-i} + \Psi^{i,-i}) (\log(M_t/M_{t+k}) + p_t^{i*}) + (\Psi^{-i} \Psi^{-i} + \Psi^{-i,-i}) (\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i}) \right\} \right].
\end{aligned} \tag{15}
$$

This equation is approximated up to the first order. It should be noted that the 0-th order term, where $\hat{p}_t = 0$, yields the steady-state price p, which is different from that under flexible prices. Because of the first-order approximation, we can ignore second-order terms such as $p_t^{i*}\hat{p}_{t+k'}^{-i}$.

Note that we have

$$\begin{aligned}\mathbb{E}_t[\log(M_{t+k}/M_t)] &= \sum_{k'=1}^{k} \mathbb{E}_t\varepsilon_{t+k'} = \sum_{k'=1}^{k} \rho^{k'}\varepsilon_t \\ &= \rho(1-\rho^k)/(1-\rho)\cdot\varepsilon_t \text{ for } k \geq 1,\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\log(M_{t+k}/M_t) &= \sum_{k=1}^{\infty}\theta^k\beta^k\left\{\rho(1-\rho^k)/(1-\rho)\cdot\varepsilon_t\right\} \\ &= \frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t,\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^{2k+1}\beta^k\mathbb{E}_t\log(M_{t+k}/M_{t-1}) &= \sum_{k=0}^{\infty}\theta^{2k+1}\beta^k\left\{(1-\rho^{k+1})/(1-\rho)\cdot\varepsilon_t\right\} \\ &= \frac{1}{1-\rho}\theta\left(\frac{1}{1-\theta^2\beta} - \frac{\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t,\end{aligned}$$

$$\begin{aligned}\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'} &= (1-\theta)\theta^k\frac{1-1/\theta^{k+1}}{1-1/\theta} \\ &= -\theta^{k+1}(1-1/\theta^{k+1}) \\ &= 1-\theta^{k+1},\end{aligned}$$

$$\begin{aligned}\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'} &= (1-\theta)\theta^k\frac{1/\theta-1/\theta^{k+1}}{1-1/\theta} \\ &= -\theta^{k+1}(1/\theta-1/\theta^{k+1}) \\ &= 1-\theta^k,\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\right] &= \sum_{k=0}^{\infty}\theta^{2k}\beta^k(1-\theta)\left[\sum_{k'=0}^{k}\theta^{-k'}\right] \\ &= \sum_{k=0}^{\infty}\theta^{2k}\beta^k(1-\theta)\frac{1-1/\theta^{k+1}}{1-1/\theta} \\ &= \frac{1-\theta}{1-1/\theta}\sum_{k=0}^{\infty}\theta^{2k}\beta^k(1-1/\theta^{k+1}) \\ &= -\theta\left[\frac{1}{1-\theta^2\beta} - \frac{1/\theta}{1-\theta\beta}\right] \\ &= \frac{-\theta}{1-\theta^2\beta} + \frac{1}{1-\theta\beta},\end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\right] &= \sum_{k=1}^{\infty}\theta^{2k}\beta^k(1-\theta)\left[\sum_{k'=1}^{k}\theta^{-k'}\right] \\
&= \sum_{k=1}^{\infty}\theta^{2k}\beta^k(1-\theta)\frac{1/\theta-1/\theta^{k+1}}{1-1/\theta} \\
&= \frac{1-\theta}{1-1/\theta}\sum_{k=0}^{\infty}\theta^{2k}\beta^k(1/\theta-1/\theta^{k+1}) \\
&= -\theta\left[\frac{1/\theta}{1-\theta^2\beta}-\frac{1/\theta}{1-\theta\beta}\right] \\
&= \frac{-1}{1-\theta^2\beta}+\frac{1}{1-\theta\beta},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\log(M_{t+k'}/M_{t+k})\right] \\
&=\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\sum_{k''=k'+1}^{k}(-\varepsilon_{t+k''})\right] \\
&=\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\sum_{k''=k'+1}^{k}(-\rho^{k''}\varepsilon_t)\right] \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\frac{\rho^{k'+1}-\rho^{k+1}}{1-\rho}\varepsilon_t\right] \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\sum_{k'=0}^{k}\{\theta^{-k'}\rho^{k'}-\theta^{-k'}\rho^k\}\right]\rho\varepsilon_t \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{1-(\rho/\theta)^{k+1}}{1-\rho/\theta}-\frac{1-(1/\theta)^{k+1}}{1-1/\theta}\rho^k\right]\rho\varepsilon_t \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{(1-(\rho/\theta)^{k+1})(1-1/\theta)-\{1-(1/\theta)^{k+1}\}\rho^k(1-\rho/\theta)}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{1-(\rho/\theta)^{k+1}-1/\theta+\rho^{k+1}/\theta^{k+2}-\rho^k+\rho^k/\theta^{k+1}+\rho^{k+1}/\theta-\rho^{k+1}/\theta^{k+2}}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t \\
&=-\sum_{k=0}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{1-(\rho/\theta)^{k+1}-1/\theta-\rho^k+\rho^k/\theta^{k+1}+\rho^{k+1}/\theta}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t \\
&=-\sum_{k=0}^{\infty}\theta^{2k}\beta^k\frac{1-\theta}{1-\rho}\left[\frac{\rho^k/\theta^{k+1}(1-\rho)-\rho^k(1-\rho/\theta)+1-1/\theta}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t \\
&=-\frac{1-\theta}{(1-\rho)(1-\rho/\theta)(1-1/\theta)}\left[\frac{(1-\rho)/\theta}{1-\theta\beta\rho}-\frac{1-\rho/\theta}{1-\theta^2\beta\rho}+\frac{1-1/\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&=\frac{1}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho}-\frac{\theta-\rho}{1-\theta^2\beta\rho}-\frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\log(M_{t+k'}/M_{t+k})\right]\\
&=-\sum_{k=1}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\sum_{k'=1}^{k}\{\theta^{-k'}\rho^{k'}-\theta^{-k'}\rho^k\}\right]\rho\varepsilon_t\\
&=-\sum_{k=1}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{\rho/\theta-(\rho/\theta)^{k+1}}{1-\rho/\theta}-\frac{1/\theta-(1/\theta)^{k+1}}{1-1/\theta}\rho^k\right]\rho\varepsilon_t\\
&=-\sum_{k=1}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{(\rho/\theta-(\rho/\theta)^{k+1})(1-1/\theta)-\{1/\theta-(1/\theta)^{k+1}\}\rho^k(1-\rho/\theta)}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t\\
&=-\sum_{k=1}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{\rho/\theta-(\rho/\theta)^{k+1}-\rho/\theta^2+\rho^{k+1}/\theta^{k+2}-\rho^k/\theta+\rho^k/\theta^{k+1}+\rho^{k+1}/\theta^2-\rho^{k+1}/\theta^{k+2}}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t\\
&=-\sum_{k=1}^{\infty}\theta^k\beta^k\frac{1-\theta}{1-\rho}\theta^k\left[\frac{\rho/\theta-(\rho/\theta)^{k+1}-\rho/\theta^2-\rho^k/\theta+\rho^k/\theta^{k+1}+\rho^{k+1}/\theta^2}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t\\
&=-\sum_{k=1}^{\infty}\theta^{2k}\beta^k\frac{1-\theta}{1-\rho}\left[\frac{\rho^k/\theta^{k+1}(1-\rho)-\rho^k/\theta(1-\rho/\theta)+\rho/\theta(1-1/\theta)}{(1-\rho/\theta)(1-1/\theta)}\right]\rho\varepsilon_t\\
&=-\frac{1-\theta}{(1-\rho)(1-\rho/\theta)(1-1/\theta)}\left[\frac{(1-\rho)\beta\rho}{1-\theta\beta\rho}-\frac{(1-\rho/\theta)\theta\beta\rho}{1-\theta^2\beta\rho}+\frac{\theta\beta\rho(1-1/\theta)}{1-\theta^2\beta}\right]\rho\varepsilon_t\\
&=\frac{\theta\beta\rho}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho}-\frac{\theta-\rho}{1-\theta^2\beta\rho}-\frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_t\begin{pmatrix}p^{*i}_{t+k'}\\ p^{*-i}_{t+k'}\\ \varepsilon_{t+k'+1}\end{pmatrix}&=\mathbb{E}_t\begin{pmatrix}\Gamma&\Gamma^*&\Gamma^\varepsilon\\ \Gamma^*&\Gamma&\Gamma^\varepsilon\\ 0&0&\rho\end{pmatrix}\begin{pmatrix}p^{*i}_{t+k'-1}\\ p^{*-i}_{t+k'-1}\\ \varepsilon_{t+k'}\end{pmatrix}\\
&=\begin{pmatrix}\Gamma&\Gamma^*&\Gamma^\varepsilon\\ \Gamma^*&\Gamma&\Gamma^\varepsilon\\ 0&0&\rho\end{pmatrix}^{k'+1}\begin{pmatrix}\hat{p}^{i}_{t-1}\\ \hat{p}^{-i}_{t-1}\\ \varepsilon_t\end{pmatrix}\\
&\equiv\mathbb{\Gamma}^{k'+1}\begin{pmatrix}\hat{p}^{i}_{t-1}\\ \hat{p}^{-i}_{t-1}\\ \varepsilon_t\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\theta^k\beta^k\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\mathbb{\Gamma}^{k'+1}\right]\\
&=\sum_{k=0}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\left[\sum_{k'=0}^{k}\theta^{-k'}\mathbb{\Gamma}^{k'+1}\right]\\
&=\sum_{k=0}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\left[I-(\mathbb{\Gamma}/\theta)^{k+1}\right]\\
&=(1-\theta)\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\sum_{k=0}^{\infty}\theta^{2k}\beta^k\left[I-(\mathbb{\Gamma}/\theta)^{k+1}\right]\\
&=(1-\theta)\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\left[\frac{1}{1-\theta^2\beta}I-\mathbb{\Gamma}/\theta\left[I-(\theta\beta\mathbb{\Gamma})\right]^{-1}\right],
\end{aligned}
$$

thus

$$\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\begin{pmatrix}p^{*i}_{t+k'}\\ p^{*-i}_{t+k'}\\ \varepsilon_{t+k'+1}\end{pmatrix}\right]$$

$$=(1-\theta)\Gamma\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\left[\frac{1}{1-\theta^2\beta}I-\mathbb{\Gamma}/\theta\left[I-(\theta\beta\mathbb{\Gamma})\right]^{-1}\right]\begin{pmatrix}\hat{p}^{i}_{t-1}\\ \hat{p}^{-i}_{t-1}\\ \varepsilon_t\end{pmatrix}$$

$$\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}p^{*-i}_{t+k'}\right]\equiv\mathbb{\Gamma}^{i}_{k0}\hat{p}^{i}_{t-1}+\mathbb{\Gamma}^{-i}_{k0}\hat{p}^{-i}_{t-1}+\mathbb{\Gamma}^{\varepsilon}_{k0}\varepsilon_t.$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\theta^k\beta^k\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\mathbb{\Gamma}^{k'+1}\right]\\
=&\sum_{k=1}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\mathbb{\Gamma}\left[\sum_{k'=1}^{k}\theta^{-k'}\mathbb{\Gamma}^{k'}\right]\\
=&\sum_{k=1}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\left[\mathbb{\Gamma}/\theta-(\mathbb{\Gamma}/\theta)^{k+1}\right]\\
=&(1-\theta)\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\mathbb{\Gamma}/\theta\sum_{k=1}^{\infty}\theta^{2k}\beta^k\left[I-(\mathbb{\Gamma}/\theta)^k\right]\\
=&(1-\theta)\mathbb{\Gamma}\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\mathbb{\Gamma}/\theta\left[\frac{\theta^2\beta}{1-\theta^2\beta}I-\theta\beta\mathbb{\Gamma}\left[I-(\theta\beta\mathbb{\Gamma})\right]^{-1}\right],\end{aligned}$$

thus

$$\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}\begin{pmatrix}p^{*i}_{t+k'}\\ p^{*-i}_{t+k'}\\ \varepsilon_{t+k'+1}\end{pmatrix}\right]$$

$$=(1-\theta)\Gamma\left[I-(\mathbb{\Gamma}/\theta)\right]^{-1}\mathbb{\Gamma}/\theta\left[\frac{\theta^2\beta}{1-\theta^2\beta}I-\theta\beta\mathbb{\Gamma}\left[I-(\theta\beta\mathbb{\Gamma})\right]^{-1}\right]\begin{pmatrix}\hat{p}^{i}_{t-1}\\ \hat{p}^{-i}_{t-1}\\ \varepsilon_t\end{pmatrix}$$

$$\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=1}^{k}(1-\theta)\theta^{k-k'}p^{*-i}_{t+k'}\right]\equiv\mathbb{\Gamma}^{i}_{k1}\hat{p}^{i}_{t-1}+\mathbb{\Gamma}^{-i}_{k1}\hat{p}^{-i}_{t-1}+\mathbb{\Gamma}^{\varepsilon}_{k1}\varepsilon_t.$$

Thus, equation (15) is rearranged as

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[1 + \Psi^i(\log(M_t/M_{t+k}) + p_t^{i*})\right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left[\Psi^{-i}(\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i})\right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left\{\Psi^{-i}(\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i})\right\}\right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \\
&\cdot \mathbb{E}_t \left\{(1-\frac{1}{p})\Psi^i + \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)(\log(M_t/M_{t+k}) + p_t^{i*}) + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i})\right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k (1-\frac{1}{p})\Psi^i \sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right) \sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} (\log(M_t/M_{t+k}) + p_t^{i*}) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k (1-\frac{1}{p}) \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} (\Psi^i\Psi^{-i} + \Psi^{i,-i})(\log(M_{t+k'}/M_{t+k}) + \hat{p}_{t+k'}^{-i})\right] \\
&+ (1-\frac{1}{p}) \left(\frac{-1}{1-\theta^2\beta} + \frac{1}{1-\theta\beta}\right) \Psi^{-i}\Gamma^* \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k (1-\theta^k) \mathbb{E}_t \left\{\left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\{\log(M_t/M_{t+k}) + p_t^{i*}\}\right\}\Gamma^* \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t (1-\frac{1}{p})\Gamma^* \left[\sum_{k'=1}^{k} (1-\theta)\theta^{k-k'} \left\{(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i})\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[1 + \Psi^i(\log(M_t/M_{t+k}) + p_t^{i*})\right] \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t \left[\Psi^{-i}(\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i})\right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left\{\Psi^{-i}(\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i})\right\}\right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left\{(1-\frac{1}{p})\Psi^i + \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)(\log(M_t/M_{t+k}) + p_t^{i*})\right\} \\
&+ \sum_{k=0}^{\infty} \theta^{2k+1} \beta^k \mathbb{E}_t (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\log(M_{t-1}/M_{t+k}) + \hat{p}_{t-1}^{-i}) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k (1-\frac{1}{p}) \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} (\Psi^i\Psi^{-i} + \Psi^{i,-i})(\log(M_{t+k'}/M_{t+k}) + \hat{p}_{t+k'}^{-i})\right] \\
&+ (1-\frac{1}{p}) \left(\frac{-1}{1-\theta^2\beta} + \frac{1}{1-\theta\beta}\right) \Psi^{-i}\Gamma^* \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k (1-\theta^k) \mathbb{E}_t \left\{\left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\{\log(M_t/M_{t+k}) + p_t^{i*}\}\right\}\Gamma^* \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t (1-\frac{1}{p})\Gamma^* \left[\sum_{k'=1}^{k} (1-\theta)\theta^{k-k'} \left\{(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\log(M_{t+k'}/M_{t+k}) + p_{t+k'}^{*-i})\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 =& \frac{1}{1-\theta\beta} \\
&- \Psi^i \frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t + \Psi^i \frac{1}{1-\theta\beta}\left(\Gamma \hat{p}^i_{t-1} + \Gamma^* \hat{p}^{-i}_{t-1} + \Gamma^\varepsilon \varepsilon_t\right) \\
&- \Psi^{-i} \frac{1}{1-\rho}\theta\left(\frac{1}{1-\theta^2\beta} - \frac{\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t + \Psi^{-i}\frac{\theta}{1-\theta^2\beta}\hat{p}^{-i}_{t-1} \\
&+ \Psi^{-i}\frac{1}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ \Psi^{-i}\left(\mathbb{\Gamma}^i_{k0}\hat{p}^i_{t-1} + \mathbb{\Gamma}^{-i}_{k0}\hat{p}^{-i}_{t-1} + \mathbb{\Gamma}^\varepsilon_{k0}\varepsilon_t\right) \\
&+ (1-\frac{1}{p})\Psi^i \frac{1}{1-\theta\beta} \\
&- \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t \\
&+ \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{1}{1-\theta\beta}\left(\Gamma\hat{p}^i_{t-1} + \Gamma^*\hat{p}^{-i}_{t-1} + \Gamma^\varepsilon\varepsilon_t\right) \\
&- (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{1}{1-\rho}\theta\left(\frac{1}{1-\theta^2\beta} - \frac{\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{\theta}{1-\theta^2\beta}\hat{p}^{-i}_{t-1} \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{1}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\mathbb{\Gamma}^i_{k0}\hat{p}^i_{t-1} + \mathbb{\Gamma}^{-i}_{k0}\hat{p}^{-i}_{t-1} + \mathbb{\Gamma}^\varepsilon_{k0}\varepsilon_t) \\
&+ (1-\frac{1}{p})\left(\frac{-1}{1-\theta^2\beta} + \frac{1}{1-\theta\beta}\right)\Psi^{-i}\Gamma^* \\
&- \left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\Gamma^*\left\{\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t - \frac{\rho}{1-\rho}\left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho}\right)\varepsilon_t\right\} \\
&+ \left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^2\beta}{1-\theta^2\beta}\right)\Gamma^*\left(\Gamma\hat{p}^i_{t-1} + \Gamma^*\hat{p}^{-i}_{t-1} + \Gamma^\varepsilon\varepsilon_t\right) \\
&+ (1-\frac{1}{p})\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})\frac{\theta\beta\rho}{(1-\rho)(1-\rho/\theta)}\left[\frac{1-\rho}{1-\theta\beta\rho} - \frac{\theta-\rho}{1-\theta^2\beta\rho} - \frac{1-\theta}{1-\theta^2\beta}\right]\rho\varepsilon_t \\
&+ (1-\frac{1}{p})\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\mathbb{\Gamma}^i_{k1}\hat{p}^i_{t-1} + \mathbb{\Gamma}^{-i}_{k1}\hat{p}^{-i}_{t-1} + \mathbb{\Gamma}^\varepsilon_{k1}\varepsilon_t). \qquad (16)
\end{aligned}
$$

■

Steady State: Proof of Lemma 1 From equation (14), in the steady state, we should have

$$0 = \frac{1}{1-\theta\beta} + (1-\frac{1}{p})\Psi^i\frac{1}{1-\theta\beta} + (1-\frac{1}{p})\left(\frac{-1}{1-\theta^2\beta} + \frac{1}{1-\theta\beta}\right)\Psi^{-i}\Gamma^*$$

$$0 = 1 + (1-\frac{1}{p})\Psi^i + (1-\frac{1}{p})\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}\Gamma^*$$

$$-1 = p - 1 + (p-1)\Psi^i + (p-1)\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}\Gamma^*$$

$$-1 = (p-1)\left\{1 + \Psi^i + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}\Gamma^*\right\}$$

$$p = 1 - \left\{1 + \Psi^i + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}\Gamma^*\right\}^{-1}. \qquad (17)$$

When $\Gamma^* = 0$, then

$$p = \frac{\Psi^i}{\Psi^i + 1}.$$

When $\Psi^i = -\frac{1+\tau}{\tau}$ and $\Psi^{-i} = \frac{1}{\tau}$, we have

$$\begin{aligned} p &= 1 - \left\{1 - \frac{1+\tau}{\tau} + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\frac{1}{\tau}\Gamma^*\right\}^{-1} \\ &= 1 - \left\{-\frac{1}{\tau} + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\frac{1}{\tau}\Gamma^*\right\}^{-1} \\ &= 1 + \tau\left\{1 - \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Gamma^*\right\}^{-1}. \end{aligned}$$

Log-linearization around the Steady State: Proof of Lemma 2 In equation (14), the term of $\hat{p}_{t-1}^{-i}$ equals

$$\begin{aligned} 0 =& \Psi^i \frac{1}{1-\theta\beta}\left(\Gamma^* \hat{p}_{t-1}^{-i}\right) \\ &+ \Psi^{-i}\frac{\theta}{1-\theta^2\beta}\hat{p}_{t-1}^{-i} \\ &+ \Psi^{-i}\left(\mathbb{\Gamma}_{k0}^{-i}\hat{p}_{t-1}^{-i}\right) \\ &+ \left(\frac{\Psi^i}{p} + (1-\frac{1}{p})(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{1}{1-\theta\beta}\left(\Gamma^*\hat{p}_{t-1}^{-i}\right) \\ &+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{\theta}{1-\theta^2\beta}\hat{p}_{t-1}^{-i} \\ &+ (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\mathbb{\Gamma}_{k0}^{-i}\hat{p}_{t-1}^{-i}) \\ &+ \left(\frac{1}{p}\Psi^{-i} + (1-\frac{1}{p})(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^2\beta}{1-\theta^2\beta}\right)\Gamma^*\left(\Gamma^*\hat{p}_{t-1}^{-i}\right) \\ &+ (1-\frac{1}{p})\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\mathbb{\Gamma}_{k1}^{-i}\hat{p}_{t-1}^{-i}). \end{aligned}$$

Suppose $\Psi^{-i} = 0$. Then, the term of $\hat{p}_{t-1}^{-i}$ equals $0 = \frac{1}{1-\theta\beta}\Gamma^*\left(\Psi^i(\Psi^i+1) - \Psi^{i,i}\right)$, which leads to $\Gamma^* = 0$.

By approximating as $p \simeq \frac{\Psi^i}{\Psi^i+1}$, we have the condition for Γ^* to satisfy:

$$\begin{aligned} 0 =& \Psi^i\frac{1}{1-\theta\beta}\Gamma^* \\ &+ \Psi^{-i}\frac{\theta}{1-\theta^2\beta} \\ &+ \Psi^{-i}\mathbb{\Gamma}_{k0}^{-i} \\ &+ \left(\Psi^i + 1 - \frac{1}{\Psi^i}(\Psi^i\Psi^i + \Psi^{i,i})\right)\frac{1}{1-\theta\beta}(\Gamma^*) \\ &- \frac{1}{\Psi^i}(\Psi^i\Psi^{-i} + \Psi^{i,-i})\frac{\theta}{1-\theta^2\beta} \\ &- \frac{1}{\Psi^i}(\Psi^i\Psi^{-i} + \Psi^{i,-i})(\mathbb{\Gamma}_{k0}^{-i}) \\ &+ \left(\frac{\Psi^i+1}{\Psi^i}\Psi^{-i} - \frac{1}{\Psi^i}(\Psi^i\Psi^{-i} + \Psi^{i,-i})\right)\left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^2\beta}{1-\theta^2\beta}\right)\Gamma^*(\Gamma^*) \\ &- \frac{1}{\Psi^i}\Gamma^*(\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i})(\mathbb{\Gamma}_{k1}^{-i}). \end{aligned}$$

$$
\begin{aligned}
0 =& \Psi^{i} \frac{1}{1-\theta\beta} \Gamma^{*} \\
&+ \Psi^{-i} \frac{\theta}{1-\theta^{2}\beta} \\
&+ \Psi^{-i} \mathbb{\Gamma}_{k0}^{-i} \\
&+ \left(1 - \frac{\Psi^{i,i}}{\Psi^{i}}\right) \frac{1}{1-\theta\beta} (\Gamma^{*}) \\
&- \left(\Psi^{-i} + \frac{\Psi^{i,-i}}{\Psi^{i}}\right) \frac{\theta}{1-\theta^{2}\beta} \\
&- \left(\Psi^{-i} + \frac{\Psi^{i,-i}}{\Psi^{i}}\right) (\mathbb{\Gamma}_{k0}^{-i}) \\
&+ \left(\frac{\Psi^{-i}}{\Psi^{i}} - \frac{\Psi^{i,-i}}{\Psi^{i}}\right) \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^{2}\beta}{1-\theta^{2}\beta}\right) \Gamma^{*} (\Gamma^{*}) \\
&- \frac{1}{\Psi^{i}} \Gamma^{*} (\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i}) (\mathbb{\Gamma}_{k1}^{-i}).
\end{aligned}
$$

$$
\begin{aligned}
0 =& \Psi^{i} \frac{1}{1-\theta\beta} \Gamma^{*} \\
&+ \left(1 - \frac{\Psi^{i,i}}{\Psi^{i}}\right) \frac{1}{1-\theta\beta} (\Gamma^{*}) \\
&- \frac{\Psi^{i,-i}}{\Psi^{i}} \frac{\theta}{1-\theta^{2}\beta} \\
&- \frac{\Psi^{i,-i}}{\Psi^{i}} (\mathbb{\Gamma}_{k0}^{-i}) \\
&+ \left(\frac{\Psi^{-i}}{\Psi^{i}} - \frac{\Psi^{i,-i}}{\Psi^{i}}\right) \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^{2}\beta}{1-\theta^{2}\beta}\right) \Gamma^{*} (\Gamma^{*}) \\
&- \frac{1}{\Psi^{i}} \Gamma^{*} (\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i}) (\mathbb{\Gamma}_{k1}^{-i}).
\end{aligned}
$$

$$
\begin{aligned}
(\Psi^{i,-i} - \Psi^{-i}) \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^{2}\beta}{1-\theta^{2}\beta}\right) \Gamma^{*2} + (\Psi^{-i}\Psi^{-i} + \Psi^{-i,-i}) \mathbb{\Gamma}_{k1}^{-i} \Gamma^{*} + \Psi^{i,-i} \left(\frac{\theta}{1-\theta^{2}\beta} + \mathbb{\Gamma}_{k0}^{-i}\right) \\
= \frac{1}{1-\theta\beta} \left(\Psi^{i}(\Psi^{i}+1) - \Psi^{i,i}\right) \Gamma^{*}. \qquad (18)
\end{aligned}
$$

Proof of Corollary 1 In the CES/Hotelling models, we have $(n-1)\Psi^{-i} = -(1+\Psi^{i})$. Further, we assume $\Psi^{i,i} = \Psi^{-i,-i} = -\Psi^{i,-i} = -(\Psi^{-i})^{2}$, which holds when $n = 2$. We allow a slight deviation for superelasticity $\Psi^{i,i}$ and $\Psi^{i,-i}$ as $-(\Psi^{-i})^{2} - \gamma^{i,i}\Psi^{-i}$ and $(\Psi^{-i})^{2} + \gamma^{i,-i}\Psi^{-i}$, respectively, where $|\gamma^{i,i}, \gamma^{i,-i}| \ll 1$. Then, the above equation becomes

$$
\begin{aligned}
((\Psi^{-i})^{2} + \gamma^{i,-i}\Psi^{-i} - \Psi^{-i}) \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^{2}\beta}{1-\theta^{2}\beta}\right) \Gamma^{*2} + \{(\Psi^{-i})^{2} + \gamma^{i,-i}\Psi^{-i}\} \left(\frac{\theta}{1-\theta^{2}\beta} + \mathbb{\Gamma}_{k0}^{-i}\right) \\
= \frac{1}{1-\theta\beta} \left((-1-(n-1)\Psi^{-i})(-(n-1)\Psi^{-i}) + (\Psi^{-i})^{2} + \gamma^{i,i}\Psi^{-i}\right) \Gamma^{*}.
\end{aligned}
$$

$$
(\Psi^{-i} + \gamma^{i,-i} - 1) \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^{2}\beta}{1-\theta^{2}\beta}\right) \Gamma^{*2} - \frac{1}{1-\theta\beta} \left(n - 1 + \gamma^{i,i} + \left((n-1)^{2} + 1\right) \Psi^{-i}\right) \Gamma^{*} + (\Psi^{-i} + \gamma^{i,-i}) \left(\frac{\theta}{1-\theta^{2}\beta} + \mathbb{\Gamma}_{k0}^{-i}\right) = 0
$$

Further, if $|\mathbb{\Gamma}_{k0}^{-i}|$ is sufficiently small and $\beta \simeq 1$, we have

$$
\begin{aligned}
\Gamma^{*} &= \frac{\frac{1}{1-\theta}\left(n - 1 + \gamma^{i,i} + \left((n-1)^{2}+1\right)\Psi^{-i}\right) \pm \sqrt{\left\{\frac{1}{1-\theta}\left(n - 1 + \gamma^{i,i} + ((n-1)^{2}+1)\Psi^{-i}\right)\right\}^{2} - 4(\Psi^{-i} + \gamma^{i,-i} - 1)\left(\frac{\theta}{1-\theta} - \frac{\theta^{2}}{1-\theta^{2}}\right)(\Psi^{-i} + \gamma^{i,-i})\left(\frac{\theta}{1-\theta^{2}}\right)}}{2(\Psi^{-i} - 1)\left(\frac{\theta}{1-\theta} - \frac{\theta^{2}}{1-\theta^{2}}\right)} \\
&= \frac{\frac{1}{1-\theta}\left(n - 1 + \gamma^{i,i} + \left((n-1)^{2}+1\right)\Psi^{-i}\right) \pm \sqrt{\left\{\frac{1}{1-\theta}\left(n - 1 + \gamma^{i,i} + ((n-1)^{2}+1)\Psi^{-i}\right)\right\}^{2} - 4(\Psi^{-i} + \gamma^{i,-i} - 1)\left(\frac{\theta}{1-\theta}\frac{1}{1+\theta}\right)^{2}(\Psi^{-i} + \gamma^{i,-i})}}{2(\Psi^{-i} - 1)\frac{\theta}{1-\theta}\frac{1}{1+\theta}} \\
&= \frac{n - 1 + \gamma^{i,i} + \left((n-1)^{2}+1\right)\Psi^{-i} \pm \sqrt{\left(n - 1 + \gamma^{i,i} + ((n-1)^{2}+1)\Psi^{-i}\right)^{2} - 4(\Psi^{-i} + \gamma^{i,-i} - 1)\left(\frac{\theta}{1+\theta}\right)^{2}(\Psi^{-i} + \gamma^{i,-i})}}{2(\Psi^{-i} - 1)\frac{\theta}{1+\theta}}.
\end{aligned}
$$

According to numerical simulations, the lower value in the above equation constitutes the approximate solution for Γ^*, while the higher value does not probably because $|\mathbb{\Gamma}_{k0}^{-i}|$ is no longer sufficiently small. Thus, we have

$$\Gamma^* = \frac{n-1+\gamma^{i,i}+\left((n-1)^2+1\right)\Psi^{-i} - \sqrt{(n-1+\gamma^{i,i}+((n-1)^2+1)\Psi^{-i})^2 - 4(\Psi^{-i}+\gamma^{i,-i}-1)\left(\frac{\theta}{1+\theta}\right)^2(\Psi^{-i}+\gamma^{i,-i})}}{2(\Psi^{-i}-1)\frac{\theta}{1+\theta}}. \tag{19}$$

When $\theta/(1+\theta) \simeq 1/2$, we have $\Gamma^* \simeq \frac{n-1+\left((n-1)^2+1\right)\Psi^{-i}-\sqrt{(n-1+((n-1)^2+1)\Psi^{-i})^2-(\Psi^{-i}-1)(\Psi^{-i})}}{(\Psi^{-i}-1)}$. This suggests that $\Gamma^* > 0$ when $\Psi^{-i} > 1$, $\Gamma^* > 0$ when $0 < \Psi^{-i} < 1$, and $\Gamma^* < 0$ when $\Psi^{-i} < 0$ and $|\Psi^{-i}| \ll 1$. Furthermore, when $n = 2$, we can show that $\partial\Gamma^*/\partial\Psi^{-i} \simeq c\{13(\Psi^{-i})^2 - 16\Psi^{-i} + 13\} > 0$ when $\theta/(1+\theta) \simeq 1/2$, where c is a positive constant. We can also show that $\partial\Gamma^*/\partial\gamma^{i,i} < 0$ and $\partial\Gamma^*/\partial\gamma^{i,-i} > 0$.

A.4 Numerical Illustrations

We solve p, Γ, Γ^*,and Γ^ε numerically without resorting to the approximation given by Lemma 2 and Corollary 1. Figure 1 shows how the persistence of monetary policy shocks ρ influences the policy function.

To derive Corollary 1, we assumed $p \simeq /\Psi^i(\Psi^i+1)$, $|\mathbb{\Gamma}_{k0}^{-i}| \ll 1$, $\beta \simeq 1$, $(n-1)\Psi^{-i} = -(1+\Psi^i)$, and $\Psi^{i,i} = \Psi^{-i,-i} = -\Psi^{i,-i} \simeq -(\Psi^{-i})^2$. If this assumption were invalid, the difference between numerical calculation and analytical approximation would be large. We compare the coefficient of dynamic strategic complementarity Γ^* when we calculate it numerically based on Proposition 1 in the Appendix and when we approximate it based on Corollary 1. We calculate Γ^* by changing one of the following four parameter values: transport cost τ, the number of firms n, the Calvo parameter θ, and the persistence of monetary policy shocks ρ.

Figure 2 shows that the two lines are similar, suggesting that Corollary 1 provides a good approximation for Γ^*. The approximation error is large only when θ is close to one, that is, the price is highly sticky.

B Details on the Hotelling Address Model

B.1 Best Response

Firm A's profit is given by

$$\Pi(p^A, p^B) = \begin{cases} 0 & \text{if } \frac{\log p^A - \log p^B}{\tau} \geq 1 \\ (p^A - W)\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{2\tau}\right)\frac{M}{p^A} & \text{if } -1 < \frac{\log p^A - \log p^B}{\tau} < 1 \\ (p^A - W)\frac{M}{p^A} & \text{if } \frac{\log p^A - \log p^B}{\tau} \leq -1. \end{cases} \tag{20}$$

Thus, the derivative of firm A's profit with respect to p^A given firm B's price p^B is given by

$$\Pi(p^A, p^B) = \begin{cases} 0 & \text{if } \frac{\log p^A - \log p^B}{\tau} \geq 1 \\ W\frac{M}{(p^A)^2}\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{2\tau}\right) + (p^A - W)\left(-\frac{1}{2\tau p^A}\right)\frac{M}{p^A} & \text{if } -1 < \frac{\log p^A - \log p^B}{\tau} < 1 \\ W\frac{M}{(p^A)^2} & \text{if } \frac{\log p^A - \log p^B}{\tau} \leq -1. \end{cases}$$

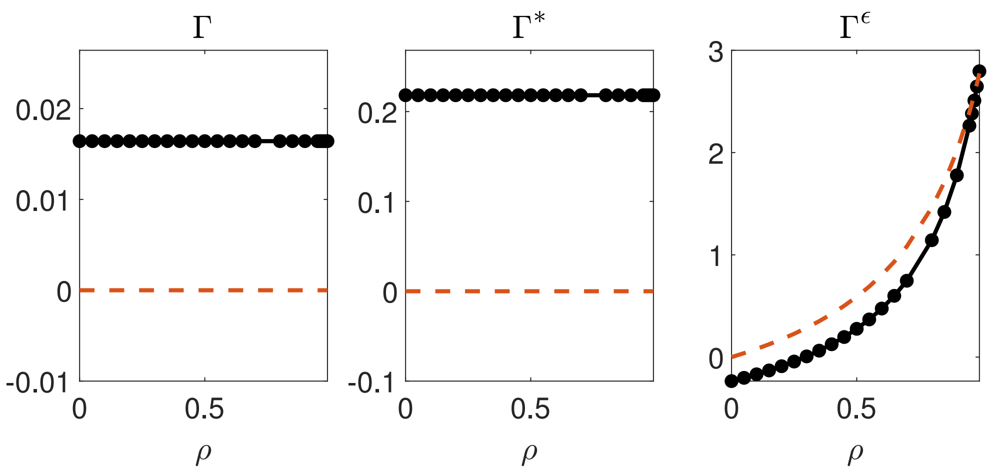

Figure 1: Policy Functions under Price Stickiness: Dependence on the Persistence of Monetary Policy Shocks

Note: The figure shows the coefficients of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t$. The horizontal axis represents the persistence of monetary policy shocks (ρ). The dashed line represents the coefficients of policy functions for the CES monopolistic competition model ($n \to \infty$ and $\sigma = 9$).

If $-1 < \frac{\log p^A - \log p^B}{\tau} < 1$ (the second line in equation (20)), the derivative is zero when p^A satisfies

$$W\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{2\tau}\right) = (p^A - W)\left(\frac{1}{2\tau}\right)$$
$$p^A + W \log p^A = W\left(\tau + 1 + \log p^B\right). \tag{21}$$

We define such p^A by $p^{A*}(p^B)$. Since the left-hand side of the equation increases with p^A monotonically from $-\infty$ to ∞, such $p^{A*}(p^B)$ is uniquely determined. Moreover it is clear that $p^{A*}(p^B)$ is increasing with p^B. When p^B is low, this $p^{A*}(p^B)$ falls in the range of $\frac{\log p^A - \log p^B}{\tau} \geq 1$ (the first line in equation (20)), which causes firm A to earn zero revenue and profit. In this case, the best response is arbitrary (i.e. not limited to $p^{A*}(p^B)$; $p^A = W$ is a best response as well), but we simply assume that $p^{A*}(p^B)$ is a best response. If $\frac{\log p^A - \log p^B}{\tau} \leq -1$ (the third line in equation (20)), firm A should choose as high price as possible, which equals $\exp(\log p^B - \tau)$. Note that the equality of $-1 = \frac{\log p^{A*}(p^B) - \log p^B}{\tau}$ holds when $p^A = W(2\tau + 1)$ and $p^B = \exp\left[\tau + \log\{W(2\tau+1)\}\right]$. Further, strict inequality of $-1 < \frac{\log p^{A*}(p^B) - \log p^B}{\tau}$ holds when $p^B < \exp\left[\tau + \log\{W(2\tau+1)\}\right]$.

In sum, the best response of p^A given p^B is expressed as follows:

$$p^A(p^B) = \begin{cases} p^{A*}(p^B) & \text{if } p^B < \exp\left[\tau + \log\{W(2\tau+1)\}\right] \\ \exp(\log p^B - \tau) & \text{if } p^B \geq \exp\left[\tau + \log\{W(2\tau+1)\}\right] \end{cases} \tag{22}$$

Differentiating $p^{A*}(p^B)$ with respect to p^B around the steady-state value of $p = p^A = p^B$ yields $p^{*'} + W/p^* \cdot p^{*'} = W/p^B$, which, in turn, becomes

$$p^{*'} = 1/(2+\tau). \tag{23}$$

Thus, the degree of strategic complementarity is positive.

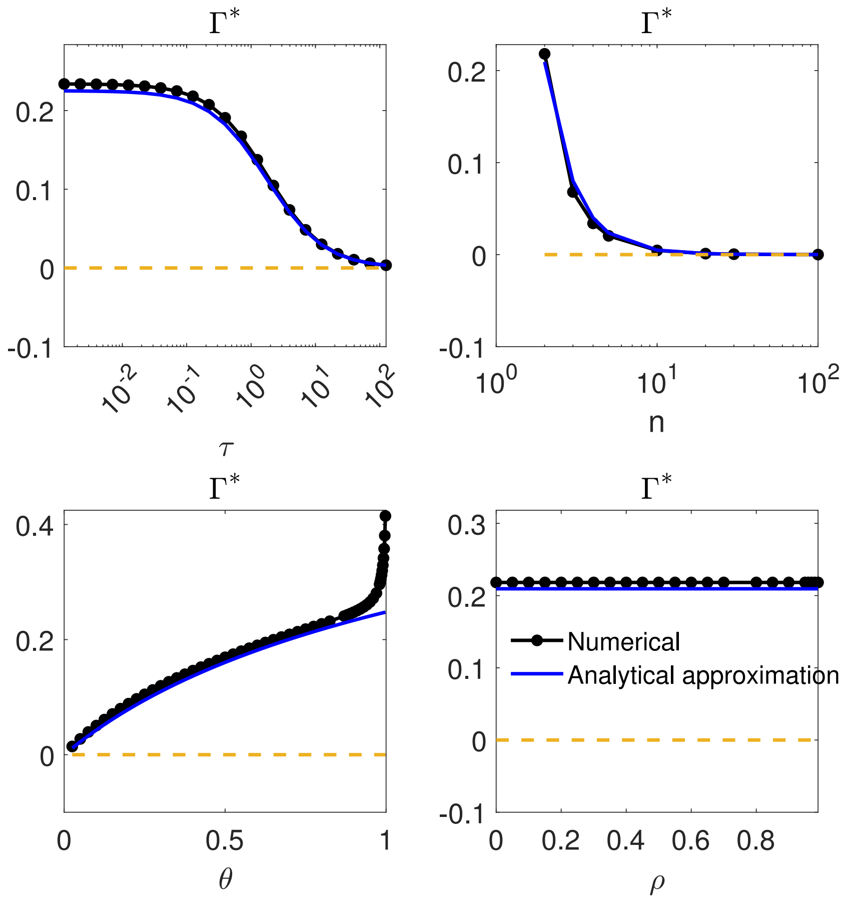

Figure 2: Dynamic Strategic Complementarity: Numerical Calculation and Analytical Approximation

Note: The figure shows the coefficient Γ^* of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t$.

B.2 Pricing under Calvo-type Price Stickiness

When firm A has a chance to set its price at t, it sets $\bar{p}_t^A$ to maximize

$$\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t-1}^B}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log\bar{p}_{t+k'}^B}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}. \tag{24}$$

Hereafter, we assume $W_0 = M_0 = 1$ in the initial period. The first-order condition for the optimal $\bar{p}_t^A$ is given by

$$0 = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}_t^A} \right)^2 M_{t+k} \left[\theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t-1}^B}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}_t^A} \right)^2 M_{t+k} \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log\bar{p}_{t+k'}^B}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \left(-\frac{1}{2\tau\bar{p}_t^A} \right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \frac{\partial\log\bar{p}_{t+k'}^B / \partial\log\bar{p}_t^A}{2\tau\bar{p}_t^A} \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.$$

In log-linearization, let us denote $\bar{p}_t^A \equiv pM_t e^{p_t^{A*}}$, $p_t^B \equiv pM_t e^{\hat{p}_t^B}$ as well as $\partial\log\bar{p}_{t+k}^B/\partial\log\bar{p}_t^A \equiv \Gamma^*$ for any $k \geq 1$ (which we will define in detail later; it is independent of k because we consider a case in which the price of firm A is unchanged at $\bar{p}_t^A$). Note that $\partial\log\bar{p}_t^B/\partial\log\bar{p}_t^A = 0$ because firm B does not know at t that firm A revises its price at t. Then, the log-linearization leads to

$$0 = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{pe^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[\theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{pe^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{pe^{p_t^{A*}}}\right) \left(-\frac{1}{2\tau} \right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{pe^{p_t^{A*}}}\right) (1-\theta^k) \frac{\Gamma^*}{2\tau} \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.$$

Steady State In the steady state, it becomes

$$0 = \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{1}{p} \right) \cdot \frac{1}{2} + \sum_{k=0}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p} \right) \cdot \left(-\frac{1}{2\tau} \right)$$
$$+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p} \right) (1-\theta^k) \left(\frac{\Gamma^*}{2\tau} \right),$$

which yields

$$p = 1 + \tau \left(1 - \frac{(1-\theta)\theta\beta}{1-\theta^2\beta} \Gamma^* \right)^{-1}. \tag{25}$$

Log-linearization The log-linearization proceeds as

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[\theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p + p p_t^{A*} - M_{t+k}/M_t \right) \left(-\frac{1}{2\tau} \right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) (1-\theta^k) \frac{\Gamma^*}{2\tau} \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}
$$

The term $\frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$ equals one because $P_t C_t = M_t$. Thus, we have

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left(\frac{1}{2} \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) \left(-\frac{1}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t)(1-\theta^k) \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(p p_t^{A*} \right) \left(-\frac{1}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(p p_t^{A*} \right) (1-\theta^k) \left(\frac{\Gamma^*}{2\tau} \right) \qquad (26) \\
=& A_t + B_t, \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
A_t \equiv& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left(\frac{1}{2} \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) \left(-\frac{1}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t)(1-\theta^k) \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(p p_t^{A*} \right) \left(-\frac{1}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(p p_t^{A*} \right) (1-\theta^k) \left(\frac{\Gamma^*}{2\tau} \right),
\end{aligned}
$$

$$
B_t \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right].
$$

Note that for $k \geq 1$, we have

$$\mathbb{E}_t[\log(M_{t+k}/M_t)] = \sum_{k'=1}^{k} \mathbb{E}_t \varepsilon_{t+k'} = \rho(1-\rho^k)/(1-\rho) \cdot \varepsilon_t,$$

$$\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) = \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t.$$

As for A_t, we have

$$\begin{aligned}
A_t =& \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t \left(\frac{1}{2} \right) \\
&+ (-1) \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t \left(-\frac{1}{2\tau} \right) \\
&+ (-1) \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t \left(\frac{\Gamma^*}{2\tau} \right) \\
&- (-1) \frac{\rho}{1-\rho} \left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho} \right) \varepsilon_t \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \frac{\theta}{1-\theta^2\beta} (p_t^{A*} - \hat{p}_{t-1}^B + \varepsilon_t) \left(-\frac{1}{2\tau} \right) \\
&+ \frac{1}{1-\theta\beta} pp_t^{A*} \left(-\frac{1}{2\tau} \right) \\
&+ \frac{\theta\beta}{1-\theta\beta} pp_t^{A*} \left(\frac{\Gamma^*}{2\tau} \right) - \frac{\theta^2\beta}{1-\theta^2\beta} pp_t^{A*} \left(\frac{\Gamma^*}{2\tau} \right)
\end{aligned} \tag{28}$$

As for B_t, we have

$$\begin{aligned}
B_t =& \sum_{k=0}^{\infty} \theta^k \beta^k \left[(1-\theta^{k+1}) \left(-p_t^{A*} \right) \left(\frac{1}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} p_{t+k'}^{B*} \left(\frac{1}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left[\sum_{k'=0}^{k} (1-\theta) \theta^{k-k'} \left(\frac{\rho(1-\rho^{k'}) \cdot \varepsilon_t}{1-\rho} \right) \left(\frac{1}{2\tau} \right) \right] \\
=& \left(\frac{1}{1-\theta\beta} - \frac{\theta}{1-\theta^2\beta} \right) \left(-p_t^{A*} \right) \left(\frac{1}{2\tau} \right) \\
&+ B_t' \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left[1 - \theta^{k+1} - \frac{\rho^{k+1} - \theta^{k+1}}{\rho - \theta} (1-\theta) \right] \frac{\rho\varepsilon_t}{1-\rho} \left(\frac{1}{2\tau} \right) \\
=& B_t' \\
&+ \left(-p_t^{A*} \right) \left(\frac{1}{1-\theta\beta} - \frac{\theta}{1-\theta^2\beta} \right) \left(\frac{1}{2\tau} \right) \\
&+ \left[\frac{1}{1-\theta\beta} - \frac{1-\theta}{\rho-\theta} \frac{\rho}{1-\theta\beta\rho} + \frac{1-\rho}{\rho-\theta} \frac{\theta}{1-\theta^2\beta} \right] \frac{\rho\varepsilon_t}{1-\rho} \left(\frac{1}{2\tau} \right)
\end{aligned} \tag{29}$$

if $\rho \neq \theta$, where

$$\begin{aligned}
B'_t &= \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} p^{B*}_{t+k'} \right] \left(\frac{1}{2\tau} \right) \\
\mathbb{E}_t \left[B'_{t+1} \right] &= \sum_{k=1}^{\infty} \theta^{k-1} \beta^{k-1} \mathbb{E}_t \left[\sum_{k'=1}^{k} (1-\theta)\theta^{k-k'} p^{B*}_{t+k'} \right] \left(\frac{1}{2\tau} \right) \\
B'_t &= \theta\beta \mathbb{E}_t \left[B'_{t+1} \right] + \sum_{k=0}^{\infty} \theta^k \beta^k (1-\theta)\theta^k \mathbb{E}_t \left[p^{B*}_t \right] \left(\frac{1}{2\tau} \right) \\
&= \theta\beta \mathbb{E}_t \left[B'_{t+1} \right] + \frac{1-\theta}{1-\theta^2\beta} \mathbb{E}_t \left[p^{B*}_t \right] \left(\frac{1}{2\tau} \right). \qquad (30)
\end{aligned}$$

We further have

$$\begin{aligned}
B'_t &= \Lambda^B \hat{p}^A_{t-1} + \Lambda^{B*} \hat{p}^B_{t-1} + \Lambda^{B\varepsilon} \varepsilon_t \qquad (31) \\
\mathbb{E}_t \left[B'_{t+1} \right] &= \mathbb{E}_t \left[\Lambda^B \hat{p}^A_t + \Lambda^{B*} \hat{p}^B_t + \Lambda^{B\varepsilon} \varepsilon_{t+1} \right] \\
&= \Lambda^B p^{A*}_t + \Lambda^{B*} \left\{ \theta \left(\hat{p}^B_{t-1} - \varepsilon_t \right) + (1-\theta) p^{B*}_t \right\} + \Lambda^{B\varepsilon} \rho \varepsilon_t \\
&= \Lambda^B \left(\Gamma \hat{p}^A_{t-1} + \Gamma^* \hat{p}^B_{t-1} + \Gamma^\varepsilon \varepsilon_t \right) \\
&+ \Lambda^{B*} \left\{ \theta \left(\hat{p}^B_{t-1} - \varepsilon_t \right) + (1-\theta) \left(\Gamma \hat{p}^B_{t-1} + \Gamma^* \hat{p}^A_{t-1} + \Gamma^\varepsilon \varepsilon_t \right) \right\} \\
&+ \Lambda^{B\varepsilon} \rho \varepsilon_t \\
&= \left(\Lambda^B \Gamma + (1-\theta) \Lambda^{B*} \Gamma^* \right) \hat{p}^A_{t-1} \\
&+ \left(\Lambda^B \Gamma^* + \theta \Lambda^{B*} + (1-\theta) \Lambda^{B*} \Gamma \right) \hat{p}^B_{t-1} \\
&+ \left(\Lambda^B \Gamma^\varepsilon - \theta \Lambda^{B*} + (1-\theta) \Lambda^{B*} \Gamma^\varepsilon + \Lambda^{B\varepsilon} \rho \right) \varepsilon_t.
\end{aligned}$$

It should be noted that the optimal prices are expressed in the following forms:

$$\begin{aligned}
p^{A*}_t &= \Gamma \hat{p}^A_{t-1} + \Gamma^* \hat{p}^B_{t-1} + \Gamma^\varepsilon \varepsilon_t \qquad (32) \\
p^{B*}_t &= \Gamma \hat{p}^B_{t-1} + \Gamma^* \hat{p}^A_{t-1} + \Gamma^\varepsilon \varepsilon_t, \qquad (33) \\
\partial \log \bar{p}^B_{t+k} / \partial \log \bar{p}^A_t &= \partial p^{B*}_{t+k} / \partial p^{A*}_t = \Gamma^*, \qquad (34)
\end{aligned}$$

where Γ, Γ^*, and Γ^ε represent coefficients to be determined. They are determined by equation (26) or (27) by comparing coefficients on $\hat{p}^A_{t-1}$, $\hat{p}^B_{t-1}$, and ε_t, each. More precisely, we use equation (25) for p and equations (28) and (29) for A_t and B_t.

Inflation Dynamics Aggregate price index is given by

$$\log P_t = \int_0^1 \log p^j_t dj \qquad (35)$$

for product line j. For each j, suppose the log-linearized prices set by firm A and B are given by $\hat{p}^A_t$ and $\hat{p}^B_t$, respectively. Then, consumers $x = \frac{1}{2} - \frac{\hat{p}^A_t - \hat{p}^B_t}{2\tau}$ buy from firm A at $\hat{p}^A_t$ and $1-x$ consumers buy from firm B at $\hat{p}^B_t$. Thus, the log-linearized price aggregated at the level of product line j is given by

$$\begin{aligned}
&x\hat{p}_t^A + (1-x)\hat{p}_t^B \\
=&\left(\frac{1}{2} - \frac{\hat{p}_t^A - \hat{p}_t^B}{2\tau}\right)\hat{p}_t^A + \left(\frac{1}{2} - \frac{\hat{p}_t^B - \hat{p}_t^A}{2\tau}\right)\hat{p}_t^B \\
=&\frac{\hat{p}_t^A + \hat{p}_t^B}{2} - \frac{\left(\hat{p}_t^A - \hat{p}_t^B\right)^2}{2\tau} \simeq \frac{\hat{p}_t^A + \hat{p}_t^B}{2}.
\end{aligned}$$

Under symmetry, the log-linearized aggregate price becomes

$$\begin{aligned}
\hat{P}_t &= \int_0^1 \left(\frac{\hat{p}_t^A + \hat{p}_t^B}{2}\right) dj \\
&= \theta \int_0^1 (\hat{p}_{t-1} - \varepsilon_t)\, dj + (1-\theta)\int_0^1 p_t^* dj \\
&= \theta \hat{P}_{t-1} - \theta\varepsilon_t + (1-\theta)\left(\Gamma\hat{p}_{t-1} + \Gamma^*\hat{p}_{t-1} + \Gamma^\varepsilon \varepsilon_t\right) \\
&= \kappa \hat{P}_{t-1} + \{(1-\theta)\Gamma^\varepsilon - \theta\}\,\varepsilon_t, \qquad (36)
\end{aligned}$$

where $\kappa \equiv \theta + (1-\theta)(\Gamma + \Gamma^*)$. For the inflation rate $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$, we obtain

$$\begin{aligned}
\pi_t - \varepsilon_t &= \kappa\left(\pi_{t-1} - \varepsilon_{t-1}\right) + \{(1-\theta)\Gamma^\varepsilon - \theta\}\left(\varepsilon_t - \varepsilon_{t-1}\right) \\
\pi_t &= \kappa\pi_{t-1} + (1-\theta)(1+\Gamma^\varepsilon)\varepsilon_t - \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}\,\varepsilon_{t-1}. \qquad (37)
\end{aligned}$$

This suggests that inflation dynamics is influenced by Γ, Γ^*, and Γ^ε, which are, in turn, influenced by τ.

The above equation can be further transformed into

$$\pi_t = (\kappa+\rho)\pi_{t-1} - \kappa\rho\pi_{t-2} + (1-\theta)(1+\Gamma^\varepsilon)\mu_t - \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}\,\mu_{t-1}. \qquad (38)$$

Then, we obtain

$$\mu_t = \left(\pi_t - (\kappa+\rho)\pi_{t-1} + \kappa\rho\pi_{t-2}\right)/(1-\theta)/(1+\Gamma^\varepsilon) + \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}/(1-\theta)/(1+\Gamma^\varepsilon)\cdot\mu_{t-1}$$

$$\begin{aligned}
\pi_t &= (\kappa+\rho)\pi_{t-1} - \kappa\rho\pi_{t-2} + (1-\theta)(1+\Gamma^\varepsilon)\mu_t \\
&\quad - \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}\left\{\left(\pi_{t-1} - (\kappa+\rho)\pi_{t-2} + \kappa\rho\pi_{t-3}\right)/(1-\theta)/(1+\Gamma^\varepsilon) + \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}/(1-\theta)/(1+\Gamma^\varepsilon)\cdot\mu_{t-2}\right\} \\
&= (1-\theta)(1+\Gamma^\varepsilon)\mu_t + \left(\kappa + \rho - \frac{\kappa + (1-\theta)\Gamma^\varepsilon - \theta}{(1-\theta)(1+\Gamma^\varepsilon)}\right)\pi_{t-1} + O(\pi_{t-2}), \qquad (39)
\end{aligned}$$

where $O(\pi_{t-2})$ represents the term consisting of π_{t-2-j} for $j = 0, 1, \cdots$.

Aggregate Output Aggregate output is given by $Y_t = M_t/P_t$. The log-linearization yields

$$\hat{Y}_t = -\hat{P}_t. \qquad (40)$$

Welfare Welfare is expressed as

$$\begin{aligned}
U &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - (L_t + \tau D_t)\right] \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(M_t/P_t) - M_t/P_t - \tau D_t\right].
\end{aligned}$$

The first and second terms in welfare are approximated up to the second order as

$$\begin{aligned}\log(M_t/P_t) - M_t/P_t &= \log[(1+\tau)^{-1}e^{-\hat{P}_t}] - (1+\tau)^{-1}e^{-\hat{P}_t} \\ &= -\log(1+\tau) - (1+\tau)^{-1} - \hat{P}_t + (1+\tau)^{-1}(\hat{P}_t - \hat{P}_t^2/2) \\ &= -\log(1+\tau) - \frac{1}{1+\tau} - \frac{\tau}{1+\tau}\hat{P}_t - \frac{1/2}{1+\tau}\hat{P}_t^2. \end{aligned} \tag{41}$$

The third term in welfare, shopping distance D_t, is approximated up to the second order as

$$\begin{aligned} D_t &= \int_0^{\frac{1}{2}-\frac{\log p_t^A - \log p_t^B}{2\tau}} x dx + \int_{\frac{1}{2}-\frac{\log p_t^A - \log p_t^B}{2\tau}}^1 (1-x)dx \\ &= \frac{\left(\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{2\tau}\right)^2}{2} + \frac{\left(\frac{1}{2} - \frac{\log p_t^B - \log p_t^A}{2\tau}\right)^2}{2} \\ &= \frac{\left(\frac{1}{2} - \frac{\hat{p}_t^A - \hat{p}_t^B}{2\tau}\right)^2}{2} + \frac{\left(\frac{1}{2} - \frac{\hat{p}_t^B - \hat{p}_t^A}{2\tau}\right)^2}{2} \\ &= \frac{1}{4} + \left(\frac{\hat{p}_t^A - \hat{p}_t^B}{2\tau}\right)^2. \end{aligned} \tag{42}$$

C Pricing under Rotemberg-type Price Stickiness

Hotelling Model We assume the Hotelling address model. Firms pay Rotemberg-type price adjustment costs given by $\Theta/2\,(p_t/p_{t-1}-1)^2 M_t$ when they set p_t, where Θ represents a cost parameter. This makes a firm's optimal price depend on its own price in the previous period. Moreover, the firm's optimal price depends on the rival's price in the previous period, because the rival's price in the previous period influences the rival's price in the current period and, in turn, the firm's profit. In what follows, we log-linearize equilibrium around steady state so that log-linearized price $\hat{p}_t$ is given by $p_t \equiv pM_te^{\hat{p}_t}$ and the optimal log-linearized price is denoted by p_t^*.

Given the Markov perfect equilibrium, the log-linearized optimal prices are expressed in the following policy functions:

$$p_t^{A*} = \Gamma\hat{p}_{t-1}^A + \Gamma^*\hat{p}_{t-1}^B + \Gamma^\varepsilon\varepsilon_t \tag{43}$$

$$p_t^{B*} = \Gamma\hat{p}_{t-1}^B + \Gamma^*\hat{p}_{t-1}^A + \Gamma^\varepsilon\varepsilon_t. \tag{44}$$

The coefficients Γ and Γ^* show the elasticity of the optimal price in the current period with respect to a change in its own and the rival's price, respectively, in the previous period. In equilibrium, we should have $\hat{p}_t = p_t^*$. The coefficient Γ^ε shows the elasticity of the optimal price with respect to the money supply shock.

In the presence of the Rotemberg-type price adjustment costs, firm A sets p_{t+k}^A $(k = 0, 1, \cdots)$ to maximize

$$\max \sum_{k=0}^{\infty} \mathbb{E}_t\beta^k \frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\left\{\left(1-\frac{W_{t+k}}{p_{t+k}^A}\right)\left(\frac{1}{2}-\frac{\log p_{t+k}^A - \log p_{t+k}^B}{2\tau}\right) - \frac{\Theta}{2}\left(\frac{p_{t+k}^A}{p_{t+k-1}^A}-1\right)^2\right\}\frac{M_{t+k}}{M_t}, \tag{45}$$

where Λ_t represents the stochastic discount factor given by C_t^{-1}. The first-order condition for the optimal p_t^A is given by

$$
\begin{aligned}
0 = & \left(\frac{1}{p_t^A}\right)^2 M_t \left(\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{2\tau}\right) + \left(1 - \frac{M_t}{p_t^A}\right)\left(-\frac{1}{2\tau p_t^A}\right) \\
& -\Theta\left(\frac{p_t^A}{p_{t-1}^A} - 1\right)\frac{1}{p_{t-1}^A} \\
& +\beta\mathbb{E}_t\left[\kappa\left(\frac{p_{t+1}^A}{p_t^A} - 1\right)\frac{p_{t+1}^A}{(p_t^A)^2} \cdot \frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}\right] \\
& +\mathbb{E}_t\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}\left\{\left(1 - \frac{M_{t+1}}{p_{t+1}^A}\right) \cdot \frac{\partial \log p_{t+1}^B / \partial \log p_t^A}{2\tau p_t^A}\right\}\frac{M_{t+1}}{M_t},
\end{aligned}
\tag{46}
$$

where

$$
\partial \log p_{t+1}^B / \partial \log p_t^A = \partial \hat{p}_{t+1}^{B*} / \partial \hat{p}_t^A = \Gamma^*.
$$

Firm A has to take account of how its price influences the rival firm B's price, which is given by $\partial \log p_{t+1}^B / \partial \log p_t^A$. Hereafter, we assume $W_0 = M_0 = 1$ in the initial period.

Steady State In equilibrium, symmetry yields $\hat{p}_t = \hat{p}_t^A = \hat{p}_t^B = \hat{P}_t$, where $\hat{P}_t$ represents the log-linearized aggregate price. In the steady state, equation (46) becomes

$$
0 = \left(\frac{1}{p}\right)^2 \frac{1}{2} + \left(1 - \frac{1}{p}\right)\left(-\frac{1}{2\tau p}\right) + \beta\left\{\left(1 - \frac{1}{p}\right) \cdot \frac{\Gamma^*}{2\tau p}\right\}.
$$

Thus, the steady-state optimal price is obtained as

$$
p = 1 + \frac{\tau}{1 - \beta\Gamma^*}. \tag{47}
$$

Log-linearization In log-linearization, the first-order condition (46) leads to

$$
\begin{aligned}
0 = & \frac{1}{p_t^A} M_t \left(\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{2\tau}\right) + \left(1 - \frac{M_t}{p_t^A}\right)\left(-\frac{1}{2\tau}\right) \\
& -\Theta\left(\frac{p_t^A}{p_{t-1}^A} - 1\right)\frac{p_t^A}{p_{t-1}^A} \\
& +\beta\mathbb{E}_t\left[\kappa\left(\frac{p_{t+1}^A}{p_t^A} - 1\right)\frac{p_{t+1}^A}{p_t^A} \cdot \frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}\right] \\
& +\beta\mathbb{E}_t\left[\left(1 - \frac{M_{t+1}}{p_{t+1}^A}\right) \cdot \frac{\Gamma^*}{2\tau}\frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}\right],
\end{aligned}
$$

The term $\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$ equals one because $P_t C_t = M_t$, so that we have

$$
\begin{aligned}
0 = & \frac{1}{pM_t e^{p_t^{A*}}} M_t \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}_t^B}{2\tau}\right) + \left(1 - \frac{M_t}{pM_t e^{p_t^{A*}}}\right)\left(-\frac{1}{2\tau}\right) \\
& -\Theta\left(\frac{pM_t e^{p_t^{A*}}}{pM_{t-1}e^{\hat{p}_{t-1}^A}} - 1\right)\frac{pM_t e^{p_t^{A*}}}{pM_{t-1}e^{\hat{p}_{t-1}^A}} \\
& +\beta\mathbb{E}_t\left[\Theta\left(\frac{pM_{t+1}e^{p_{t+1}^{A*}}}{pM_t e^{p_t^{A*}}} - 1\right)\frac{pM_{t+1}e^{p_{t+1}^{A*}}}{pM_t e^{p_t^{A*}}} \cdot \frac{\Lambda_{t+1}}{\Lambda_t}\frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}\right] \\
& +\beta\mathbb{E}_t\left[\left(1 - \frac{M_{t+1}}{pM_{t+1}e^{p_{t+1}^{A*}}}\right) \cdot \frac{\Gamma^*}{2\tau}\right].
\end{aligned}
$$

$$
\begin{aligned}
0 =& -\frac{p_t^{A*}}{p}\frac{1}{2} - \frac{1}{p}\frac{p_t^{A*} - p_t^{B*}}{2\tau} - \frac{p_t^{A*}}{p}\frac{1}{2\tau} \\
&-\Theta\left(\varepsilon_t + p_t^{A*} - \hat{p}_{t-1}^{A}\right) \\
&+\beta\Theta\mathbb{E}_t\left[\varepsilon_{t+1} + p_{t+1}^{A*} - p_t^{A*}\right] \\
&+\frac{\mathbb{E}_t\left[p_{t+1}^{A*}\right]}{p}\frac{\beta\Gamma^*}{2\tau},
\end{aligned}
$$

Note

$$
\begin{aligned}
\mathbb{E}_t\left[p_{t+1}^{A*}\right] =& \mathbb{E}_t\left[\Gamma p_t^{A*} + \Gamma^* p_t^{B*} + \Gamma^{\varepsilon}\varepsilon_{t+1}\right] \\
=& \Gamma p_t^{A*} + \Gamma^*\left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right) + \rho\Gamma^{\varepsilon}\varepsilon_t.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
0 =& -\frac{p_t^{A*}}{p}\frac{1}{2} - \frac{1}{p}\frac{p_t^{A*} - \left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right)}{2\tau} - \frac{p_t^{A*}}{p}\frac{1}{2\tau} \\
&-\Theta\left(\varepsilon_t + p_t^{A*} - \hat{p}_{t-1}^{A}\right) \\
&+\beta\Theta\left(\rho\varepsilon_t - p_t^{A*}\right) \\
&+\beta\Theta\left\{\Gamma p_t^{A*} + \Gamma^*\left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right) + \rho\Gamma^{\varepsilon}\varepsilon_t\right\} \\
&+\frac{1}{p}\frac{\beta\Gamma^*}{2\tau}\left\{\Gamma p_t^{A*} + \Gamma^*\left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right) + \rho\Gamma^{\varepsilon}\varepsilon_t\right\},
\end{aligned}
$$

$$
\begin{aligned}
& p_t^{A*}\left\{\frac{1}{p}\left(\frac{1}{2} + \frac{1}{\tau}\right) + \kappa(1 + \beta - \beta\Gamma) - \frac{1}{p}\frac{\beta\Gamma\Gamma^*}{2\tau}\right\} \\
=& \frac{1}{p}\frac{\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t}{2\tau} \\
& - \Theta\left(\varepsilon_t - \hat{p}_{t-1}^{A}\right) \\
& + \beta\Theta\rho\varepsilon_t \\
& + \beta\Theta\left\{\Gamma^*\left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right) + \rho\Gamma^{\varepsilon}\varepsilon_t\right\} \\
& + \frac{1}{p}\frac{\beta\Gamma^*}{2\tau}\left\{\Gamma^*\left(\Gamma\hat{p}_{t-1}^{B} + \Gamma^*\hat{p}_{t-1}^{A} + \Gamma^{\varepsilon}\varepsilon_t\right) + \rho\Gamma^{\varepsilon}\varepsilon_t\right\} \\
=& \hat{p}_{t-1}^{A}\left\{\frac{1}{p}\frac{\Gamma^*}{2\tau} + \Theta + \beta\Theta\Gamma^{*2} + \frac{1}{p}\frac{\beta\Gamma^{*3}}{2\tau}\right\} \\
& + \hat{p}_{t-1}^{B}\left\{\frac{1}{p}\frac{\Gamma}{2\tau} + \beta\Theta\Gamma\Gamma^* + \frac{1}{p}\frac{\beta\Gamma\Gamma^{*2}}{2\tau}\right\} \\
& + \varepsilon_t\left\{\frac{1}{p}\frac{\Gamma^{\varepsilon}}{2\tau} - \Theta + \beta\Theta\rho + \beta\Theta\Gamma^{\varepsilon}(\Gamma^* + \rho) + \frac{1}{p}\frac{\beta\Gamma^*}{2\tau}\Gamma^{\varepsilon}(\Gamma^* + \rho)\right\}.
\end{aligned}
$$

which suggests

$$
\Gamma = \left\{\frac{1}{p}\left(\frac{1}{2} + \frac{1}{\tau}\right) + \Theta(1 + \beta - \beta\Gamma) - \frac{1}{p}\frac{\beta\Gamma\Gamma^*}{2\tau}\right\}^{-1}\left\{\frac{1}{p}\frac{\Gamma^*}{2\tau} + \Theta + \beta\Theta\Gamma^{*2} + \frac{1}{p}\frac{\beta\Gamma^{*3}}{2\tau}\right\} \tag{48}
$$

$$
\Gamma^* = \left\{\frac{1}{p}\left(\frac{1}{2} + \frac{1}{\tau}\right) + \Theta(1 + \beta - \beta\Gamma) - \frac{1}{p}\frac{\beta\Gamma\Gamma^*}{2\tau}\right\}^{-1}\left\{\frac{1}{p}\frac{\Gamma}{2\tau} + \beta\Theta\Gamma\Gamma^* + \frac{1}{p}\frac{\beta\Gamma\Gamma^{*2}}{2\tau}\right\} \tag{49}
$$

$$
\Gamma^{\varepsilon} = \left\{\frac{1}{p}\left(\frac{1}{2} + \frac{1}{\tau}\right) + \Theta(1 + \beta - \beta\Gamma) - \frac{1}{p}\frac{\beta\Gamma\Gamma^*}{2\tau}\right\}^{-1}\left\{\frac{1}{p}\frac{\Gamma^{\varepsilon}}{2\tau} - \Theta + \beta\Theta\rho + \beta\Theta\Gamma^{\varepsilon}(\Gamma^* + \rho) + \frac{1}{p}\frac{\beta\Gamma^*}{2\tau}\Gamma^{\varepsilon}(\Gamma^* + \rho)\right\}.
$$

The third line is rearranged as

$$
\Gamma^{\varepsilon} = -\Theta(1 - \beta\rho)\left\{\frac{1}{2p}\left(1 + \frac{1}{\tau}\right) + \Theta(1 + \beta - \beta\Gamma - \beta\Gamma^* - \beta\rho) - \frac{1}{p}\frac{\beta\Gamma^*}{2\tau}(\Gamma + \Gamma^* + \rho)\right\}^{-1}. \tag{50}
$$

Inflation Dynamics Aggregate price index is given by

$$\log P_t = \int_0^1 \log p_t^j dj \tag{51}$$

for product line j. Because symmetry between two firms holds in equilibrium, the log-linearized aggregate price becomes

$$\hat{P}_t = \hat{p}_t = (\Gamma + \Gamma^*)\hat{p}_{t-1} + \Gamma^{\varepsilon}\varepsilon_t. \tag{52}$$

For the inflation rate $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$, we obtain

$$\pi_t - \varepsilon_t = (\Gamma + \Gamma^*)(\pi_{t-1} - \varepsilon_{t-1}) + \Gamma^{\varepsilon}(\varepsilon_t - \varepsilon_{t-1}). \tag{53}$$

This suggests that inflation dynamics is influenced by Γ, Γ^*, and Γ^{ε}, which are, in turn, influenced by τ.

The above equation can be further transformed into

$$\pi_t - \rho\pi_{t-1} - \mu_t = (\Gamma + \Gamma^*)(\pi_{t-1} - \rho\pi_{t-2} - \mu_{t-1}) + \Gamma^{\varepsilon}(\mu_t - \mu_{t-1}).$$

$$\pi_t = (\Gamma + \Gamma^* + \rho)\pi_{t-1} - \rho(\Gamma + \Gamma^*)\pi_{t-2} + (1 + \Gamma^{\varepsilon})\mu_t - (\Gamma + \Gamma^* + \Gamma^{\varepsilon})\mu_{t-1}. \tag{54}$$

Aggregate Output Aggregate output is given by $C_t = M_t/P_t$ and $Y_t = C_t + \Theta/2\,(P_t/P_{t-1} - 1)^2 M_t$. The log-linearization yields

$$\hat{Y}_t = -\hat{P}_t. \tag{55}$$

C.1 Comparison with a New Keynesian Model

Consumption is aggregated following the CES form of aggregation:

$$C_t = \left\{\int_0^1 C_t(j)^{\frac{\sigma-1}{\sigma}} dj\right\}^{\frac{\sigma}{\sigma-1}}. \tag{56}$$

This yields demand and price index given by $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\sigma} Y_t$ and $P_t = \left\{\int_0^1 P_t(j)^{1-\sigma} dj\right\}^{\frac{1}{1-\sigma}}$, respectively, where $C_t(j) = Y_t(j)$.

Pricing under Price Stickiness Under Rotemberg-type price stickiness, firm j sets p_{t+k} to maximize

$$\max \sum_{k=0}^{\infty} \mathbb{E}_t \beta^k \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \left(p_{t+k} Y_{t+k}(j) - W_{t+k} Y_{t+k}(j) - \frac{\Theta}{2} (p_{t+k}/p_{t+k-1} - 1)^2 M_{t+k}\right).$$

The first-order condition with respect to p_t leads to

$$\begin{aligned} 0 &= Y_t p_t^{-\sigma-1} P_t^{\sigma} \left[(1-\sigma) p_t + \sigma M_t\right] - \Theta \left(p_t/p_{t-1} - 1\right)/p_{t-1} M_t \\ &+ \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{P_t}{P_{t+1}} \Theta \left(p_{t+1}/p_t - 1\right)(p_{t+1}/p_t^2) M_{t+1}\right]. \end{aligned}$$

In equilibrium, $p_t = P_t$. Fuhrer, using log-linearization and denoting $p_t \equiv \frac{\sigma}{\sigma-1} M_t e^{p_t^*}$, we have

$$
\begin{aligned}
0 &= Y_t \left[(1-\sigma) P_t + \sigma M_t\right] - \Theta \left(\frac{M_t e^{p_t^*}}{M_{t-1} e^{\hat{p}_{t-1}}} - 1 \right) \frac{M_t e^{p_t^*}}{M_{t-1} e^{\hat{p}_{t-1}}} M_t \\
&+ \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{P_t}{P_{t+1}} \Theta \left(\frac{M_{t+1} e^{p_{t+1}^*}}{M_t e^{p_t^*}} - 1 \right) \frac{M_{t+1} e^{p_{t+1}^*}}{M_t e^{p_t^*}} M_{t+1} \right].
\end{aligned}
$$

$$
0 = -(\sigma-1) p_t^* - \Theta(\varepsilon_t + p_t^* - \hat{p}_{t-1}) + \beta \Theta \mathbb{E}_t (\varepsilon_{t+1} + p_{t+1}^* - p_t^*).
$$

$$
p_t^* \left(\sigma - 1 + \Theta + \beta\Theta\right) = \Theta \hat{p}_{t-1} + \beta \Theta \mathbb{E}_t p_{t+1}^* - \Theta(1-\beta\rho)\varepsilon_t.
$$

The solution for this equation is given by

$$
p_t^* = \Gamma^{NK} \hat{p}_{t-1} + \Gamma^{NK\varepsilon} \varepsilon_t.
$$

$$
\begin{aligned}
& p_t^* \left(\sigma - 1 + \Theta + \beta\Theta\right) \\
=\ & \Theta \hat{p}_{t-1} + \beta \Theta \mathbb{E}_t \left[\Gamma^{NK} p_t^* + \Gamma^{NK\varepsilon} \varepsilon_{t+1} \right] - \Theta(1-\beta\rho)\varepsilon_t \\
=\ & \Theta \hat{p}_{t-1} + \beta \Theta \Gamma^{NK} p_t^* + \beta\rho\Theta \Gamma^{NK\varepsilon} \varepsilon_t - \Theta(1-\beta\rho)\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
& p_t^* \left(\sigma - 1 + \Theta + \beta\Theta - \beta\Theta\Gamma^{NK}\right) \\
=\ & \Theta \hat{p}_{t-1} + \beta\rho\Theta \Gamma^{NK\varepsilon} \varepsilon_t - \Theta(1-\beta\rho)\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{NK} &= \left(\sigma - 1 + \Theta + \beta\Theta - \beta\Theta\Gamma^{NK}\right)^{-1} \Theta \\
\Gamma^{NK\varepsilon} &= \left(\sigma - 1 + \Theta + \beta\Theta - \beta\Theta\Gamma^{NK}\right)^{-1} \Theta \left(\beta\rho\Gamma^{NK\varepsilon} - 1 + \beta\rho\right) \quad .
\end{aligned}
$$

The aggregate inflation rate, $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$, is given by

$$
\pi_t = (\Gamma^{NK} + \rho)\pi_{t-1} - \rho\Gamma^{NK}\pi_{t-2} + (1 + \Gamma^{NK\varepsilon})\mu_t - (\Gamma^{NK} + \Gamma^{NK\varepsilon})\mu_{t-1}. \tag{57}
$$

Simulation Results A time unit is a quarter. In the benchmark, we normalize $W = 1$. We set transport cost $\tau = 0.125$, consistent with $\sigma = 9$, as assumed in Gali (2015). Price stickiness Θ is set at 100, so that impulse responses shown below are comparable to those based on the Calvo-type price stickiness model with $\theta = 0.75$. We also use $\rho = 0.85$ and $\beta = 0.99$.

Figure 3 shows policy functions for the optimal reset price, represented by the coefficients Γ, Γ^*, and Γ^ε. The horizontal axis represents transport cost τ in a log scale. For comparison, we plot policy functions in the standard New Keynesian model, where $\sigma = 1 + 1/\tau$ so that the steady-state markup under flexible prices is the same. Figure 4 shows policy functions for the optimal reset price, where the horizontal axis represents price stickiness Θ. Figure 5 shows the steady-state price under sticky prices. Finally, Figure 6 shows the impulse response functions to a positive money supply shock ($\mu_t = 1$ at $t = 1$) for aggregate inflation rate π_t and output $\hat{Y}_t$.

D Mixed-Strategy Pricing under Duopolistic Competition and Nominal Stickiness

We assume two firms ($i = A, B$). We consider an arbitrary invertible demand system $x_t^i = x^i(p_t^i, p_t^{-i}; M_t) = x^i(p_t^i/M_t, p_t^{-i}/M_t)$ for $i = A, B$. Firm profit is given by $\Pi_t^i = (p_t^i - W_t) x^i(p_t^i/M_t, p_t^{-i}/M_t)$.

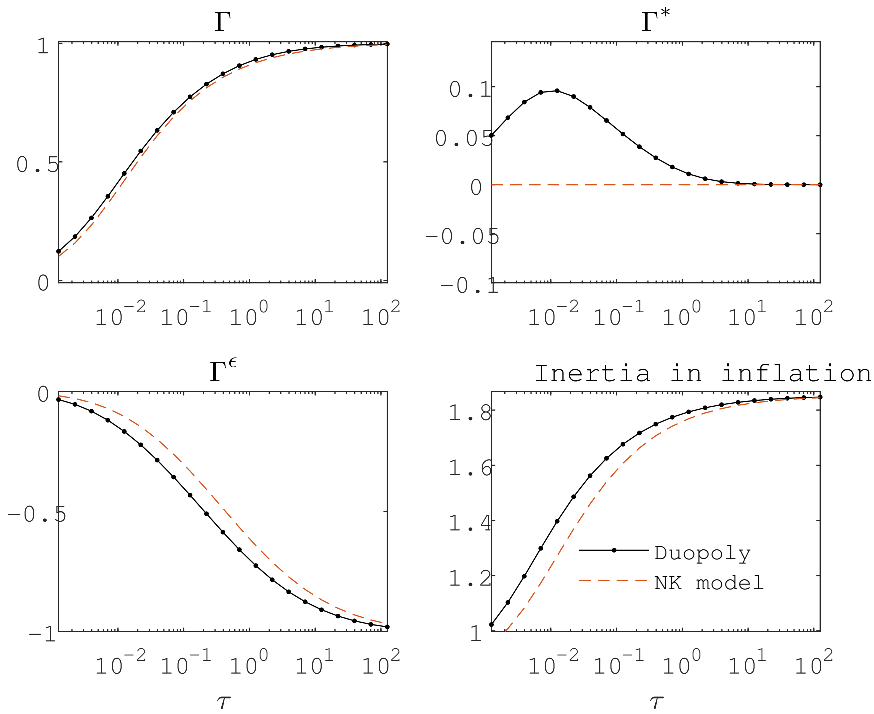

Figure 3: Policy Functions under Rotemberg-type Price Stickiness: Dependence on Transport Costs

Note: The figure shows the coefficients of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma\hat{p}_{t-1}^A + \Gamma^*\hat{p}_{t-1}^B + \Gamma^\varepsilon\varepsilon_t$. The lower right-hand panel shows the coefficient on past inflation (π_{t-1}) for the equation of inflation (π_t). The horizontal axis represents transport cost (τ, log scale).

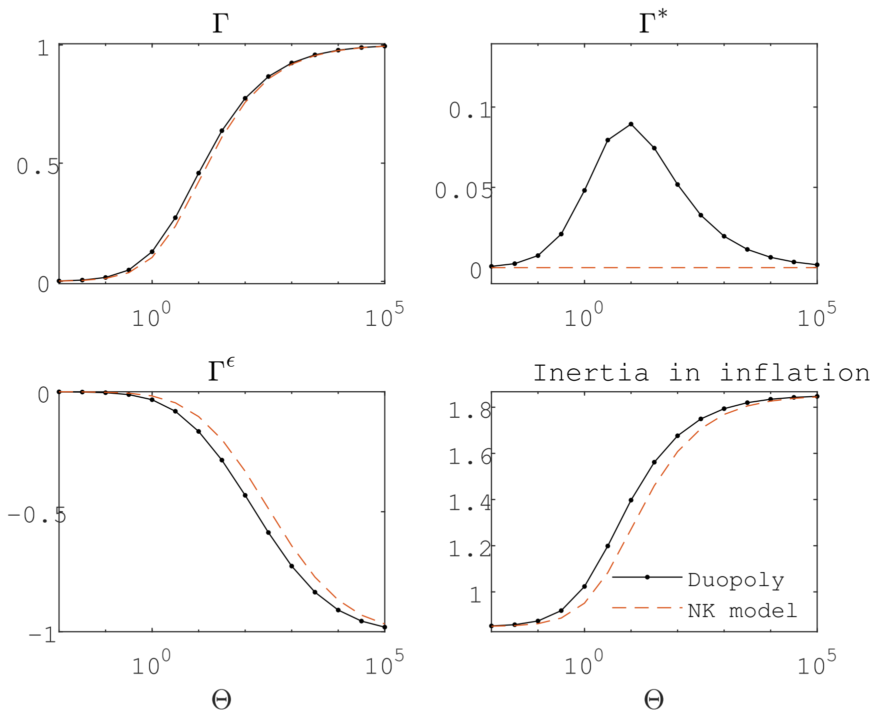

Figure 4: Policy Functions under Rotemberg-type Price Stickiness: Dependence on Price Stickiness
Note: The figure shows the coefficients of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma\hat{p}_{t-1}^{A} + \Gamma^{*}\hat{p}_{t-1}^{B} + \Gamma^{\varepsilon}\varepsilon_t$. The lower right-hand panel shows the coefficient on past inflation (π_{t-1}) for the equation of inflation (π_t). The horizontal axis represents price stickiness (Θ).

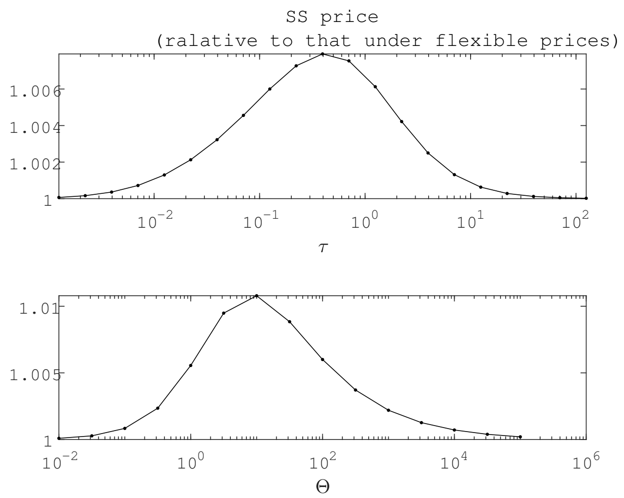

Figure 5: Steady-State Price under Rotemberg-type Price Stickiness
Note: The vertical axis represents the ratio of the steady-state price under sticky prices to that under flexible prices. The horizontal axis represents transport cost (τ, log scale) and price stickiness (Θ) in the upper and lower panels, respectively.

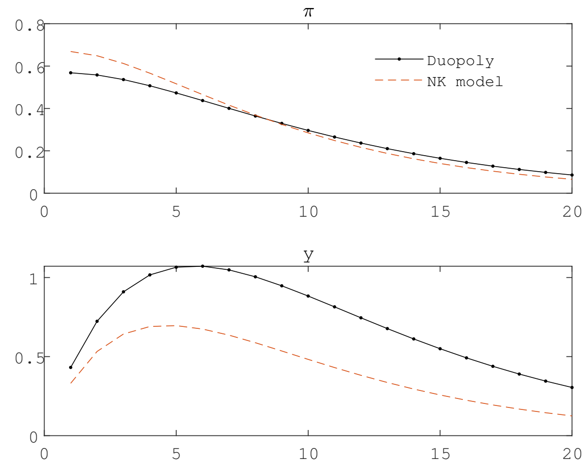

Figure 6: Impulse Responses to Money Supply Shock under Rotemberg-type Price Stickiness
Note: The horizontal axis represents quarters after a positive money supply shock occurs at $t = 1$.

Further, we assume that demand elasticities depend on prices to allow firms to set two possible prices, higher p_H (with the probability of $1-s$) and lower p_L (with the probability of s) . Aggregate nominal demand equals M_t so that $p_t^i x_t^i + p_t^{-i} x_t^{-i} = M_t$. It should be noted that kinked demand is not related to superelasticity, which is the second-order derivative of demand. Superelasticity does not influence the steady state price without price stickiness. Expected demand from choosing p is given by $(1-s)x(p, p_H) + sx(p, p_L)$. We define demand elasticity as follows:

$$\Psi^i(p_H, p_H) \equiv \frac{\partial \log x^i(p_H^i/M, p_H^{-i}/M)}{\partial \log(p^{i,H}/M)}$$
$$\Psi^i(p_H, p_L) \equiv \frac{\partial \log x^i(p_H^i/M, p_L^{-i}/M)}{\partial \log(p_L^i/M)}$$
$$\Psi^i(p_L, p_H) \equiv \frac{\partial \log x^i(p_L^i/M, p_H^{-i}/M)}{\partial \log(p_L^i/M)}$$
$$\Psi^i(p_L, p_L) \equiv \frac{\partial \log x^i(p_L^i/M, p_L^{-i}/M)}{\partial \log(p_L^i/M)}$$
$$\Psi^{-i}(p_H, p_H) \equiv \frac{\partial \log x^i(p_H^i/M, p_H^{-i}/M)}{\partial \log(p_H^{-i}/M)}.$$

D.1 Steady State without Price Stickiness

We consider the equilibrium in a steady state with $W = M = 1$. Suppose mixed strategy given by $p_H > p_L$. The first-order condition with respect to p_H yields

$$
\begin{aligned}
0 = \frac{\partial \Pi^i}{\partial p^H} &= x(p_H, p^{-i}) + (p_H - 1)\frac{\partial x(p_H, p^{-i})}{\partial p_H} \\
&= (1-s)x(p_H, p_H) + sx(p_H, p_L) \\
&+ (p_H - 1)\left\{(1-s)\frac{\partial x(p_H, p_H)}{\partial p_H} + s\frac{\partial x(p_H, p_L)}{\partial p_H}\right\} \\
&= (1-s)x(p_H, p_H) + sx(p_H, p_L) \\
&+ (p_H - 1)\left\{(1-s)\frac{x(p_H, p_H)\partial \log x(p_H, p_H)}{p_H \partial \log p_H} + s\frac{x(p_H, p_L)\partial \log x(p_H, p_L)}{p_H \partial \log p_H}\right\} \\
&= (1-s)\left\{x(p_H, p_H) + (p_H - 1)\frac{x(p_H, p_H)\Psi^i(p_H, p_H)}{p_H}\right\} \\
&+ s\left\{x(p_H, p_L) + (p_H - 1)\frac{x(p_H, p_L)\Psi^i(p_H, p_L)}{p_H}\right\} \\
0 &= (1-s)x^{HH}\left\{1 + (p_H - 1)\frac{\Psi^i(p_H, p_H)}{p_H}\right\} + sx^{HL}\left\{1 + (p_H - 1)\frac{\Psi^i(p_H, p_L)}{p_H}\right\},
\end{aligned}
\tag{58}
$$

where $x^{ij} \equiv x(p_i, p_j)$ for $i, j = H$ or L.

The first-order condition with respect to p_L yields

$$
\begin{aligned}
0 = \frac{\partial \Pi^i}{\partial p_L} &= x(p_L, p^{-i}) + (p_L - 1)\frac{\partial x(p_L, p^{-i})}{\partial p_L} \\
&= x(p_L, p^{-i}) + (p_L - 1)\frac{x(p_L, p^{-i})\partial \log x(p_L, p^{-i})}{p_L \partial \log p_L} \\
&= (1-s)\left\{x(p_L, p_H) + (p_L - 1)\frac{x(p_L, p_H)\Psi^i(p_L, p_H)}{p_L}\right\} \\
&+ s\left\{x(p_L, p_L) + (p_L - 1)\frac{x(p_L, p_L)\Psi^i(p_L, p_L)}{p_L}\right\} \\
0 &= (1-s)x^{LH}\left\{1 + (p_L - 1)\frac{\Psi^i(p_L, p_H)}{p_L}\right\} + sx^{LL}\left\{1 + (p_L - 1)\frac{\Psi^i(p_L, p_L)}{p_L}\right\}.
\end{aligned}
\tag{59}
$$

Furthermore, we should have indifference of profits choosing the higher price, $\Pi_H(p_H, p_L, s) \equiv \Pi(p_H|p_H, p_L, s)$ and $\Pi_L(p_H, p_L, s) \equiv \Pi(p_L|p_H, p_L, s)$:

$$
\begin{aligned}
(p_H - 1)x(p_H, p^{-i}) &= (p_L - 1)x(p_L, p^{-i}) \\
\Pi_H(p_H, p_L, s) &\equiv (p_H - 1)\left\{(1-s)x^{HH} + sx^{HL}\right\} \\
&= (p_L - 1)\left\{(1-s)x^{LH} + sx^{LL}\right\} \equiv \Pi_L(p_H, p_L, s).
\end{aligned}
\tag{60}
$$

Aggregate nominal demand equals M so that

$$
\begin{aligned}
2p_H x^{HH} &= 1 \\
p_H x^{HL} + p_L x^{LH} &= 1 \\
2p_L x^{LL} &= 1.
\end{aligned}
\tag{61}
$$

Proof of Corollary 2 Rewrite the first-order conditions with respect to p_H and p_L:

$$
\begin{aligned}
0 &= (1-s)x^{HH}\left\{1 + (p_H - 1)\frac{\Psi^i(p_H, p_H)}{p_H}\right\} + sx^{HL}\left\{1 + (p_H - 1)\frac{\Psi^i(p_H, p_L)}{p_H}\right\}, \\
0 &= (1-s)x^{LH}\left\{1 + (p_L - 1)\frac{\Psi^i(p_L, p_H)}{p_L}\right\} + sx^{LL}\left\{1 + (p_L - 1)\frac{\Psi^i(p_L, p_L)}{p_L}\right\}.
\end{aligned}
$$

The two equations are identical when $p_H = p_L$ on the condition that $\Psi^i(p_H, p_H) = \Psi^i(p_L, p_H)$ and $\Psi^i(p_H, p_L) = \Psi^i(p_L, p_L)$.

Next, suppose $\Psi^i(p_H, p_H) = \Psi^i(p_H, p_L)$, $\Psi^i(p_L, p_H) = \Psi^i(p_L, p_L)$, $x^{HL} = x^{HH}$, and $x^{LH} = x^{LL}$. Then, we obtain $p_H = \Psi^i(p_H, p_H)/(\Psi^i(p_H, p_H) + 1)$ and $p_L = \Psi^i(p_L, p_L)/(\Psi^i(p_L, p_L) + 1)$. Then, the profit indifference condition becomes

$$(p_H - 1)\left\{(1-s)x^{HH} + sx^{HL}\right\} = (p_L - 1)\left\{(1-s)x^{LH} + sx^{LL}\right\}$$

$$\begin{aligned}(p_H - 1)\{(1-s) + s\}\frac{1}{2p_H} &= (p_L - 1)\{(1-s) + s\}\frac{1}{2p_L} \\ -\frac{1}{2}\frac{1}{\Psi^i(p_H, p_H)} &= -\frac{1}{2}\frac{1}{\Psi^i(p_L, p_L)},\end{aligned}$$

which shows that we must have $\Psi^i(p_H, p_H) = \Psi^i(p_L, p_L)$. In this case, $p_H = p_L$.

Proof of Corollary 3 Equation (58) is rewritten as

$$\frac{p_H - 1}{p_H} = -\frac{(1-s)x^{HH} + sx^{HL}}{(1-s)x^{HH}\Psi^i(p_H, p_H) + sx^{HL}\Psi^i(p_H, p_L)}$$

By partially differentiating this with respect to p_L, we have

$$\begin{aligned}C\frac{\partial p_H}{\partial p_L} &= -\partial x^{HL}/\partial p_L\left((1-s)x^{HH}\Psi^i(p_H, p_H) + sx^{HL}\Psi^i(p_H, p_L)\right) \\ &+ s\left((1-s)x^{HH} + sx^{HL}\right)\left(\partial x^{HL}/\partial p_L \cdot \Psi^i(p_H, p_L) + x^{HL}\partial\Psi^i(p_H, p_L)/\partial p_L\right) \\ &= -x^{HL}\Psi^{-i}(p_H, p_L)/p_L\left((1-s)x^{HH}\Psi^i(p_H, p_H) + sx^{HL}\Psi^i(p_H, p_L)\right) \\ &+ s\left((1-s)x^{HH} + sx^{HL}\right)\left(-x^{HL}\Psi^{-i}(p_H, p_L)/p_L \cdot \Psi^i(p_H, p_L) + x^{HL}\Psi^{i,-i}(p_H, p_L)/p_L\right)\end{aligned}$$

$$\begin{aligned}\frac{Cp_L}{x_{HL}}\frac{\partial p_H}{\partial p_L} &= \Psi^{-i}(p_H, p_L)\left((1-s)x^{HH}(-\Psi^i(p_H, p_H)) + sx^{HL}(-\Psi^i(p_H, p_L))\right) \\ &+ s\left((1-s)x^{HH} + sx^{HL}\right)\Psi^{-i}(p_H, p_L)(-\Psi^i(p_H, p_L)) \\ &- s\left((1-s)x^{HH} + sx^{HL}\right)\Psi^{i,-i}(p_H, p_L),\end{aligned}$$

which suggests $\partial p_H/\partial p_L > 0$ unless $\Psi^{i,-i}(p_H, p_L)$ is too large.

Next, we have

$$\begin{aligned}\frac{\partial \Pi_H(p_H, p_L, s)}{\partial s} &= (p_H - 1)\left\{-x^{HH} + x^{HL}\right\} < 0 \\ \frac{\partial \Pi_L(p_H, p_L, s)}{\partial s} &= (p_L - 1)\left\{-x^{LH} + x^{LL}\right\} < 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \Pi_H(p_H, p_L, s)}{\partial s} - \frac{\partial \Pi_L(p_H, p_L, s)}{\partial s} &= (p_H - 1)\left\{-x^{HH} + x^{HL}\right\} - (p_L - 1)\left\{-x^{LH} + x^{LL}\right\} \\ &= -p_H x^{HH} + p_H x^{HL} + x^{HH} - x^{HL} \\ &+ p_L x^{LH} - p_L x^{LL} - x^{LH} + x^{LL} \\ &= -1/2 + 1 - 1/2 + x^{HH} - x^{HL} - x^{LH} + x^{LL} \\ &= (x^{LL} - x^{HL}) - (x^{LH} - x^{HH}).\end{aligned}$$

■

CES When the elasticity is constant ($\Psi^i(p_H, p_H) = \Psi^i(p_H, p_L) = \Psi^i(p_L, p_H) = \Psi^i(p_L, p_L) = \Psi$), we can show that $p_H = p_L = \frac{\Psi}{\Psi+1}$. A mixed strategy equilibrium does not arise. Even in the presence of consumer heterogeneity, aggregate demand elasticity (the sum of demand of each consumer) is constant if demand elasticity for each type of consumers is constant as in the case of CES preference. That is,

$$\mathbb{E}\left[\Psi^i\right] \equiv \frac{\partial \log x^i(p^i/M, p^{-i}/M)}{\partial \log(p^i/M)},$$

which is independent of price p^i and p^{-i}. Note that this is not the case in the Hotelling model.

Hotelling Address Model Suppose the Hotelling model with heterogeneous τ. The demand of consumer τ is given by $x_t^i(p_t^i/M_t, p_t^{-i}/M_t) = A\left(\frac{1}{2} - \frac{\log(p_t^i/M_t) - \log(p_t^{-i}/M_t)}{2\tau}\right)\frac{M_t}{p_t^i}$. Thus, we have

$$x^{HL} = (1-\alpha)\left(\frac{1}{2} - \frac{\log(p_H) - \log(p_L)}{2\tau_H}\right)\frac{1}{p_H}$$

$$x^{LH} = \alpha\frac{1}{p_L} + (1-\alpha)\left(\frac{1}{2} - \frac{\log(p_L) - \log(p_H)}{2\tau_H}\right)\frac{1}{p_L}$$

$$\begin{aligned}\Psi^i(p_H, p_H) &= \left(\frac{1}{2} - \frac{\log(p_H) - \log(p_H)}{2\tau}\right)^{-1}\mathbb{E}\left(-\frac{1}{2\tau}\right) - 1\\ &= -\mathbb{E}\left[\frac{1}{\tau}\right] - 1\end{aligned}$$

$$\begin{aligned}\Psi^i(p_H, p_L) &= \left(\frac{1}{2} - \frac{\log(p_H) - \log(p_L)}{2\tau_H}\right)^{-1}\left(-\frac{1}{2\tau_H}\right) - 1\\ &= -\frac{1-\alpha}{2p_H x^{HL}\tau_H} - 1\end{aligned}$$

$$\begin{aligned}\Psi^i(p_L, p_L) &= \left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log(p_L) - \log(p_H)}{2\tau_H}\right)\right)^{-1}\left(-\frac{1-\alpha}{2\tau_H}\right) - 1\\ &= -\frac{1-\alpha}{2p_L x^{LH}\tau_H} - 1\end{aligned}$$

$$\begin{aligned}\Psi^i(p_L, p_L) &= \left(\frac{1}{2} - \frac{\log(p_L) - \log(p_L)}{2\tau}\right)^{-1}\mathbb{E}\left(-\frac{1}{2\tau}\right) - 1\\ &= -\mathbb{E}\left[\frac{1}{\tau}\right] - 1\end{aligned}$$

$$\begin{aligned}\Psi^{-i}(p_H, p_H) &= \left(\frac{1}{2} - \frac{\log(p_H) - \log(p_H)}{2\tau}\right)^{-1}\mathbb{E}\left(-\frac{1}{2\tau}\right) - 1.\\ &= \mathbb{E}\left[\frac{1}{\tau}\right]\end{aligned}$$

$$\Psi^{-i}(p_H, p_L) = \frac{1-\alpha}{2p_H x^{HL}\tau_H}.$$

D.2 Steady State without Price Stickiness in the Hotelling Address Model

Pure Strategy The optimal price chosen by firms A and B is symmetric, $p^* = p^A = p^B$, which satisfies

$$p^* = \left\{1 + (\mathbb{E}[1/\tau])^{-1}\right\}W, \tag{62}$$

where the harmonic mean of τ is given by

$$(\mathbb{E}[1/\tau])^{-1} = \frac{1}{\alpha(1/\tau_L) + (1-\alpha)(1/\tau_H)}. \tag{63}$$

Suppose that firm A deviates to choose p^d. The pure strategy equilibrium holds if $\Pi^A(p^*, p^*) > \Pi^A(p^d, p^*)$ for any p^d. Given firm B's price p^*, firm A may be able to increase its profit by giving up revenues from price-sensitive bargain hunters and charging a higher price. In this case, the profit becomes $\Pi^A(p^d, p^*) = (1-\alpha)\left(1 - 1/p^d\right)\left(1/2 - (\log p^d - \log p^*)/(2\tau_H)\right)$ if $(\log p^d - \log p^*)/\tau_L > 1$. The deviating price p^d should satisfy $0 = 1/2 - (\log p^d - \log p^*)/(2\tau_H) - (p^d - 1)/(2\tau_H)$. The condition for pure strategy equilibrium to hold is rewritten as

$$\frac{1}{2}\left(1 - \frac{1}{p^*}\right) > (1-\alpha)\left(1 - \frac{1}{p^d}\right)\left(\frac{1}{2} - \frac{\log p^d - \log p^*}{2\tau_H}\right). \tag{64}$$

We calculate the conditions of τ_H,τ_L, and α , which must be met for the pure strategy equilibrium to exist numerically.

Mixed Strategy Suppose that firm B chooses mixed strategy, in which price is p_H^B with the probability of $1-s^B$ and p_L^B otherwise ($p_H^B > p_L^B$). Firm A also chooses mixed strategy characterized by p_H^A, p_L^A, and s^A.

When $p^A = p_H^A$, firm A's expected profit is written as

$$\begin{aligned}\Pi^A(p_H^A) = (1-s^B)\mathbb{E}&\left[\left(1 - \frac{W}{p_H^A}\right)\left(\frac{1}{2} - \frac{\log p_H^A - \log p_H^B}{2\tau}\right)\right] \\ &+ s^B(1-\alpha)\left(1 - \frac{W}{p_H^A}\right)\left(\frac{1}{2} - \frac{\log p_H^A - \log p_L^B}{2\tau_H}\right).\end{aligned}$$

If firm B sets p_L^B, firm A earns zero sales from τ_L consumers. The first-order condition with respect to p_H^A yields

$$\begin{aligned}0 = &\frac{1-s}{2}W - (1-s)\left(p_H - W\right)\mathbb{E}\left[1/2/\tau\right] \\ &+ s(1-\alpha)W\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) - s(1-\alpha)\left(p_H - W\right)\frac{1}{2\tau_H}\end{aligned} \tag{65}$$

given symmetry.

When $p^A = p_L^A$, firm A's expected profit is

$$\begin{aligned}\Pi^A(p_L^A) = (1-s^B)&\left\{\alpha\left(1 - \frac{W}{p_L^A}\right) + (1-\alpha)\left(1 - \frac{W}{p_L^A}\right)\left(\frac{1}{2} - \frac{\log p_L^A - \log p_H^B}{2\tau_H}\right)\right\} \\ &+ s^B\mathbb{E}\left[\left(1 - \frac{W}{p_L^A}\right)\left(\frac{1}{2} - \frac{\log p_L^A - \log p_L^B}{2\tau}\right)\right].\end{aligned}$$

If firm B sets p_H^B, firm A earns unit sales from τ_L consumers. The first-order condition with respect to p_L^A yields

$$\begin{aligned}0 = &(1-s)\alpha W \\ &+ (1-s)(1-\alpha)W\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right) \\ &- (1-s)(1-\alpha)\left(p_L - W\right)\frac{1}{2\tau_H} \\ &+ sW\frac{1}{2} - s\left(p_L - W\right)\mathbb{E}\left[1/2/\tau\right].\end{aligned} \tag{66}$$

Furthermore, we should have indifference of profits choosing the higher price, $\Pi_H(p_H, p_L, s) \equiv \Pi(p_H|p_H, p_L, s)$ and $\Pi_L(p_H, p_L, s) \equiv \Pi(p_L|p_H, p_L, s)$, which yields

$$\begin{aligned}\Pi_H(p_H, p_L, s) \equiv& (1-s)\left(1-\frac{W}{p_H}\right)\frac{1}{2} + s(1-\alpha)\left(1-\frac{W}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H - \log p_L}{2\tau_H}\right)\\ =&(1-s)\left\{\alpha\left(1-\frac{W}{p_L}\right) + (1-\alpha)\left(1-\frac{W}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L - \log p_H}{2\tau_H}\right)\right\}\\ &+ s\left(1-\frac{W}{p_L}\right)\frac{1}{2} \equiv \Pi_L(p_H, p_L, s). \end{aligned} \tag{67}$$

Equations (65) to (67) give the solutions for p_H, p_L, and s.

Proof of Corollary 4 The first-order condition with respect to p_H is written as

$$\begin{aligned}\frac{p_H - 1}{p_H} &= -\frac{(1-s)x^{HH} + sx^{HL}}{(1-s)x^{HH}\Psi^i(p_H, p_H) + sx^{HL}\Psi^i(p_H, p_L)}\\ &= -\frac{(1-s)x^{HH} + sx^{HL}}{(1-s)x^{HH}\left(-\mathbb{E}\left[\frac{1}{\tau}\right]-1\right) + sx^{HL}\left(-\frac{1-\alpha}{2p_H x^{HL}\tau_H}-1\right)}\\ &= \frac{(1-s)x^{HH} + sx^{HL}}{(1-s)x^{HH}\left(\mathbb{E}\left[\frac{1}{\tau}\right]+1\right) + s\frac{1-\alpha}{2p_H\tau_H} + sx^{HL}}.\end{aligned}$$

Since $\partial x^{HL}/\partial p_L > 0$, this suggests that $\partial p_H/\partial p_L > 0$.

Equation (67) follows that

$$\begin{aligned}\frac{\partial \Pi_H(p_H, p_L, s)}{\partial s} &= -\left(1-\frac{W}{p_H}\right)\frac{1}{2} + (1-\alpha)\left(1-\frac{W}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H - \log p_L}{2\tau_H}\right)\\ &= -\left(1-\frac{W}{p_H}\right)\left\{\frac{1}{2} - (1-\alpha)\left(\frac{1}{2}-\frac{\log p_H - \log p_L}{2\tau_H}\right)\right\}\\ &= -\left(1-\frac{W}{p_H}\right)\left\{\frac{1}{2}\alpha + (1-\alpha)\left(\frac{\log p_H - \log p_L}{2\tau_H}\right)\right\}\\ &< 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \Pi_L(p_H, p_L, s)}{\partial s} &= -\left\{\alpha\left(1-\frac{W}{p_L}\right) + (1-\alpha)\left(1-\frac{W}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L - \log p_H}{2\tau_H}\right)\right\} + \left(1-\frac{W}{p_L}\right)\frac{1}{2}\\ &= -\left(1-\frac{W}{p_L}\right)\left\{\alpha + (1-\alpha)\left(\frac{1}{2}-\frac{\log p_L - \log p_H}{2\tau_H}\right) - \frac{1}{2}\right\}\\ &= -\left(1-\frac{W}{p_L}\right)\left\{\frac{1}{2}\alpha + (1-\alpha)\left(\frac{\log p_H - \log p_L}{2\tau_H}\right)\right\}\\ &< 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \Pi_H(p_H, p_L, s)}{\partial s} - \frac{\partial \Pi_L(p_H, p_L, s)}{\partial s} &= -\left\{\left(1-\frac{W}{p_H}\right) - \left(1-\frac{W}{p_L}\right)\right\}\left\{\frac{1}{2}\alpha + (1-\alpha)\left(\frac{\log p_H - \log p_L}{2\tau_H}\right)\right\}\\ &= -\left(\frac{W}{p_L} - \frac{W}{p_H}\right)\left\{\frac{1}{2}\alpha + (1-\alpha)\left(\frac{\log p_H - \log p_L}{2\tau_H}\right)\right\}\\ &< 0.\end{aligned}$$

■

Welfare First, we consider the case of a pure strategy. Firms A and B set the price at p^*. Each consumer spends M/p^* for consumption C, while also supplying labor L for the same amount.

Shopping distance D equals $2\int_0^{1/2} xdx = 1/4$, but disutility from shopping differs between consumers with τ_H and τ_L. Household utility U^{pure} becomes

$$U^{pure} = \{\log(M/p^*) - (M/p^* + \mathbb{E}[\tau]/4)\}/(1-\beta). \tag{68}$$

The heterogeneity of transport costs influences the equilibrium price level. To see this, we increase deviations between τ_L and τ_H while keeping the harmonic mean of τ fixed. Specifically, we define Δ_τ so that $\tau_L = \tau \cdot (1 + \Delta_\tau/\alpha)^{-1}$ and $\tau_H = \tau \cdot (1 - \Delta_\tau/(1-\alpha))^{-1}$, which maintains $(\mathbb{E}[1/\tau])^{-1} = \tau$. Then, U^{pure} in the above equation depends on Δ_τ only through the term of $\mathbb{E}[\tau]$ because p^* is independent of Δ_τ. Moreover, we find

$$\begin{aligned}\mathbb{E}[\tau] &= (\alpha\tau_L + (1-\alpha)\tau_H) \\ &= \tau\left(\frac{\alpha}{1+\Delta_\tau/\alpha} + \frac{1-\alpha}{1-\Delta_\tau/(1-\alpha)}\right).\end{aligned}$$

The derivative of $\mathbb{E}[\tau]$ with respect to Δ_τ is

$$\tau\frac{(1-\Delta_\tau/(1-\alpha)+1+\Delta_\tau/\alpha)(1/(1-\alpha)+1/\alpha)}{(1+\Delta_\tau/\alpha)^2(1-\Delta_\tau/(1-\alpha))^2}\Delta_\tau > 0,$$

if $\Delta_\tau \ll 1$. Thus, larger deviations in transport costs increase the mean of τ, which decreases utility.

Second, we consider the case of a mixed strategy. When one firm sets p_L and the other sets p_H, all consumers with τ_L purchase from the former firm, so that $D = \int_0^1 xdx = 1/2$. As for consumers with τ_H, the fraction of $x^{HL} \equiv \frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}$ purchases from the p_L firm, and the $1 - x^{HL}$ fraction purchases from the p_H firm. Thus, $D = \int_0^{x^{HL}} xdx + \int_{x^{HL}}^1 (1-x)dx = ((x^{HL})^2 + (1-x^{HL})^2)/2$. Household utility is given by

$$\begin{aligned}U^{mixed} &= s\left\{\log(M/p^L) - M/p^L\right\}/(1-\beta) \\ &+ (1-s)\left\{\log(M/p^H) - M/p^H\right\}/(1-\beta) \\ &- \left(s^2 + (1-s)^2\right)\mathbb{E}[\tau]/4/(1-\beta) \\ &- 2s(1-s)\left\{\alpha\tau_L/2 + (1-\alpha)\tau_H((x^{HL})^2 + (1-x^{HL})^2)/2\right\}/(1-\beta).\end{aligned} \tag{69}$$

The mixed strategy has three effects on utility. First, setting the higher price p_H decreases utility by decreasing consumption. Second, setting the lower price p_L increases utility by increasing consumption. Third, the price dispersion decreases utility by increasing shopping distance.

Figure 7 shows how utility changes when τ_H changes. We fix τ_L at 0.01 and adjust α so that the harmonic mean of τ is unchanged at 0.125. The figure demonstrates that utility decreases monotonically as the harmonic-mean-preserving difference increases.

D.3 Steady State under Price Stickiness

We consider arbitrary invertible demand system, where elasticities depend on firm A and B's prices. For the case of a mixed strategy, the literature often states that regular prices are sticky while sale prices are highly flexible. As in Guimaraes and Sheedy (2011), we assume that the higher price is subject to Calvo-type price stickiness and that the lower price is perfectly flexible. Specifically, we

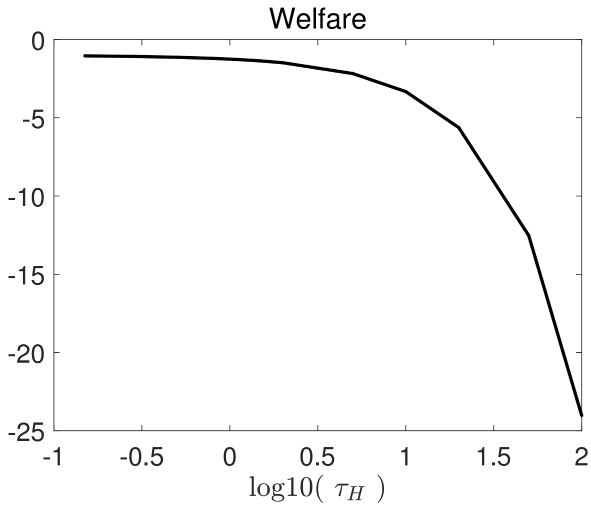

Figure 7: Household Utility under Consumers' Unobservable Heterogeneity and Flexible Prices

Note: The parameter τ_L is set at 0.01, and α is chosen to keep the harmonic mean of τ at 0.125.

assume that the lower price $p_{L,t}$ is set at $p_L W_t$, where p_L denotes the steady-state lower price when $W_0 = 1$. In other words, the lower price is indexed to the aggregate wage level (money supply) fully. This assumption is motivated by the empirical fact that firms tend to adjust the frequency of sales rather than their size to respond to shocks (Sudo et al. 2018, Kryvtsov and Vincent 2021). However, it should be noted that this price is not necessarily optimal since it does not consider price history. With the probability of $1 - \theta$, firms can revise the higher price. Firms may also choose to set the lower price, and even in this case, we assume that the rival firm can observe the higher price.

For the mixed strategy equilibrium to hold, the two choices (higher price and lower price) must yield the same payoff. However, this is very restrictive, particularly when we impose the equality of the payoff both when firms can revise their higher price and when they cannot. If the payoff from not revising the higher price is the same as that from choosing the lower price, the payoff from revising the higher price is likely to exceed that from choosing the lower price because not revising the higher price is suboptimal.

In this study, we assume that the Calvo-type lottery determines either the higher price or the frequency of sales as the variable that firms can reoptimize. With the probability of $1 - \theta$, firms can revise the higher price, and in this case, firms cannot optimize the frequency of sales, which is kept at a steady-state level (s). The equality of payoffs does not hold unless the economy is in a steady state. With the probability of θ, firms cannot revise the higher price, but they can optimize the frequency of sales s_t^n. The first-order condition with respect to the higher price is not ensured to hold. As documented by Zbaracki et al. (2004), there exist managerial costs (information gathering, decision-making, and communication costs) and customer costs (communication and negotiation costs), which prevent firms from optimizing prices. We assume that they prevent firms from optimizing both the higher price and the frequency of sales simultaneously. However, it is important to check the robustness of our results to this assumption. We will examine how a different assumption on the frequency of sales changes our results.

Proof of Lemma 3 First, we consider pure strategy. Suppose that the pure strategy equilibrium holds. The steady-state price under price stickiness equals

$$p = 1 - \left\{1 + \Psi^i(p,p) + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}(p,p)\Gamma^*\right\}^{-1}. \tag{70}$$

Proof of this Lemma 3 is the same as that for Lemma 1.

Proof of Lemma 4 Next we consider mixed strategy and calculate steady state under price stickiness. When firm i has a chance to set its price at t, it sets $\bar{p}^i_{H,t}$ to maximize

$$\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}^i_{H,t} - W_{t+k} \right) x^i(\bar{p}^i_{H,t}/M_{t+k}, p^{-i}_{t+k}/M_{t+k}) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}},$$

given that firm i chooses the higher price rather than the lower price in the periods following price revision. Regarding the probability that firm $-i$ chooses the lower price at t, s_t, we need to differentiate two cases: one is when firm $-i$ revises the higher price ($s^r_t = s$) and the other is when firm $-i$ does not (s^n_t). Then, we have

$$\begin{aligned}
\max &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}^i_{H,t} - M_{t+k} \right) \left(\theta s^n_{t+k} + (1-\theta)s\right) x^i(\bar{p}^i_{H,t}/M_{t+k}, p^{-i}_{L,t+k}/M_{t+k}) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \\
+ &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}^i_{H,t} - M_{t+k} \right) \theta(1-s^n_{t+k})\theta^k x^i(\bar{p}^i_{H,t}/M_{t+k}, p^{-i}_{H,t-1}/M_{t+k}) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \\
+ &\sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}^i_{H,t} - M_{t+k} \right) \theta(1-s^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-1-k'} x^i(\bar{p}^i_{H,t}/M_{t+k}, p^{-i}_{H,t+k'}/M_{t+k}) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \\
+ &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\bar{p}^i_{H,t} - M_{t+k} \right) (1-\theta)(1-s) x^i(\bar{p}^i_{H,t}/M_{t+k}, p^{-i}_{H,t+k}/M_{t+k}) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}}.
\end{aligned}$$

We calculate the first-order condition for the optimal $\bar{p}^i_{H,t}$ and consider its steady state as

$$\begin{aligned}
0 = &\sum_{k=0}^{\infty} \theta^k \beta^k s x^i(p_H, p_L) \\
+ &\sum_{k=0}^{\infty} \theta^k \beta^k (1-s) x^i(p_H, p_H) \\
+ &\sum_{k=0}^{\infty} \theta^k \beta^k \left(p_H - 1\right) s \frac{x^i(p_H, p_L)}{p_H} \Psi^i(p_H, p_L) \\
+ &\sum_{k=1}^{\infty} \theta^k \beta^k \left(p_H - 1\right) (1-s) \frac{x^i(p_H, p_H)}{p_H} \Psi^i(p_H, p_H) \\
+ &\sum_{k=1}^{\infty} \theta^k \beta^k \left(p_H - 1\right) (1-\theta^k)(1-s) \frac{x^i(p_H, p_H)}{p_H} \Psi^{-i}(p_H, p_H)\Gamma^* \\
+ &\sum_{k=1}^{\infty} \theta^k \beta^k \left(p_H - 1\right) \frac{\partial \log s^n_{t+k} / \partial \log \bar{p}^i_{H,t}}{p_H/s} \left[\theta x^i(p_H, p_L) - \theta^{k+1} x^i(p_H, p_H)\right] \\
- &\sum_{k=1}^{\infty} \theta^k \beta^k \left(p_H - 1\right) \frac{\partial \log s^n_{t+k} / \partial \log \bar{p}^i_{H,t}}{p_H/s} \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} x^i(p_H, p_H),
\end{aligned}$$

where we use

$$\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{\partial(\bar{p}^i_t/M_{t+k})} = \frac{x^i_{t+k}}{\bar{p}^i_t/M_{t+k}} \frac{\partial \log x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{\partial \log(\bar{p}^i_t/M_{t+k})}$$
$$= x^i \frac{\partial \log x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t-1}/M_{t+k})}{(\bar{p}^i_t/M_{t+k})\partial \log(\bar{p}^i_t/M_{t+k})}$$
$$= \frac{x^i}{p}\Psi^i,$$

$$\frac{\partial x^i(\bar{p}^i_t/M_{t+k}, p^{-i}_{t+k'}/M_{t+k})}{\partial(p^{-i}_{t+k'}/M_{t+k})} \frac{\partial p^{-i}_{t+k'}}{\partial \bar{p}^i_t} = \frac{x^i_{t+k}}{\bar{p}^{-i}_{t+k'}/M_{t+k}} \frac{\partial \log x^i(\bar{p}^i_t/M_{t+k}, \bar{p}^{-i}_{t+k'}/M_{t+k})}{\partial \log(\bar{p}^{-i}_{t+k'}/M_{t+k})} \frac{p^{-i}_{t+k'}}{\bar{p}^i_t} \frac{\partial \log p^{-i}_{t+k'}}{\partial \log \bar{p}^i_t}$$
$$= x^i \Psi^{-i} \frac{1}{p}\Gamma^*.$$

This can be rewritten as

$$0 = \frac{1}{1-\theta\beta} s x^i(p_H, p_L)$$
$$+ \frac{1}{1-\theta\beta}(1-s)x^i(p_H, p_H)$$
$$+ \frac{1}{1-\theta\beta}(p_H - 1)\, s \frac{x^i(p_H, p_L)}{p_H} \Psi^i(p_H, p_L)$$
$$+ \frac{1}{1-\theta\beta}(p_H - 1)(1-s)\frac{x^i(p_H, p_H)}{p_H}\Psi^i(p_H, p_H)$$
$$+ \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta^2\beta}{1-\theta^2\beta}\right)(p_H - 1)(1-s)\frac{x^i(p_H, p_H)}{p_H}\Psi^{-i}(p_H, p_H)\Gamma^*$$
$$+ \frac{\theta\beta}{1-\theta\beta}(p_H - 1)\frac{s\Lambda^{n*}}{p_H}\left[\theta x^i(p_H, p_L) - \theta x^i(p_H, p_H)\right]$$

$$0 = s x^i(p_H, p_L)\left(1 + \frac{p_H - 1}{p_H}\Psi^i(p_H, p_L)\right)$$
$$+ (1-s)x^i(p_H, p_H)\left(1 + \frac{p_H - 1}{p_H}\Psi^i(p_H, p_H) + \frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\frac{p_H - 1}{p_H}\Psi^{-i}(p_H, p_H)\Gamma^*\right)$$
$$+ \theta^2\beta \frac{p_H - 1}{p_H} s\Lambda^{n*}\left[x^i(p_H, p_L) - x^i(p_H, p_H)\right].$$

$$\frac{p_H - 1}{p_H}\left\{s x^i(p_H, p_L)\Psi^i(p_H, p_L) + (1-s)x^i(p_H, p_H)\Psi^i(p_H, p_H) + (1-s)\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}x^i(p_H, p_H)\Psi^{-i}(p_H, p_H)\Gamma^* + \theta^2\beta s\Lambda^{n*}\left(x^i(p\right.\right.$$
$$= -\left(s x^i(p_H, p_L) + (1-s)x^i(p_H, p_H)\right).$$

Thus, we obtain Lemma 4 as

$$\frac{p_H - 1}{p_H}\left\{s x_{HL}\Psi^i_{HL} + (1-s)/(2p_H)\Psi^i_{HH} + (1-s)/(2p_H)\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\Gamma^* + \theta^2\beta s\Lambda^{n*}\left(x^i_{HL} - 1/(2p_H)\right)\right\}$$
$$= -\left(s x_{HL} + (1-s)/(2p_H)\right). \tag{71}$$

Equation (71) is further transformed into

$$(1-s)/(2p_H)\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\Gamma^*$$
$$= -\left\{\left(s x_{HL} + (1-s)/(2p_H)\right)\frac{p_H}{p_H - 1} + s x_{HL}\Psi^i_{HL} + (1-s)/(2p_H)\Psi^i_{HH} + \theta^2\beta s\Lambda^{n*}\left(x^i_{HL} - 1/(2p_H)\right)\right\}.$$

$$\Gamma^*$$
$$= -\left\{\left(2p_H s x_{HL} + (1-s)\right)\frac{p_H}{p_H - 1} + 2p_H s x_{HL}\Psi^i_{HL} + (1-s)\Psi^i_{HH} + \theta^2\beta s\Lambda^{n*}\left(2p_H x^i_{HL} - 1\right)\right\}\left\{(1-s)\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\right\}^{-1}.$$

$$\Gamma^* = -\left\{\left(\frac{p_H}{p_H-1}+\Psi^i_{HH}\right)+\frac{s}{1-s}2p_H x_{HL}\left(\frac{p_H}{p_H-1}+\Psi^i_{HL}\right)+\frac{s}{1-s}\theta^2\beta\Lambda^{n*}\left(2p_H x^i_{HL}-1\right)\right\}\left\{\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\right\}^{-1}. \tag{72}$$

Suppose that one incorrectly evaluates Γ^* (denote by Γ^*_{pure}) from equation (70) based on pure strategy equilibrium ignoring temporary sales. Further, we assume that one uses the only higher price p_H as a proxy for p. Then, equation (70) is

$$p_H = 1-\left\{1+\Psi^i_{HH}+\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\Gamma^*_{pure}\right\}^{-1}.$$

$$\frac{1}{p_H-1} = -\left\{1+\Psi^i_{HH}+\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\Gamma^*_{pure}\right\}.$$

$$\frac{p_H}{p_H-1}+\Psi^i_{HH} = -\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\Gamma^*_{pure}.$$

$$\Gamma^*_{pure} = -\left\{\frac{p_H}{p_H-1}+\Psi^i_{HH}\right\}\left\{\frac{\theta\beta(1-\theta)}{1-\theta^2\beta}\Psi^{-i}_{HH}\right\}^{-1}.$$

Specifically, in the Hotelling model, we note

$$\begin{aligned}\Psi^i_{HH} &= -\mathbb{E}\left[\frac{1}{\tau}\right]-1\\ &= -\left(\alpha(1/\tau_L)+(1-\alpha)(1/\tau_H)\right)-1\\ \Psi^i_{HL} &= -\frac{1-\alpha}{2p_H x^{HL}\tau_H}-1\\ &= -\frac{1}{\left(1-\frac{\log(p_H)-\log(p_L)}{\tau_H}\right)\tau_H}-1\end{aligned}$$

and

$$\begin{aligned}\Psi^i_{HH}-\Psi^i_{HL} &= -\left(\alpha(1/\tau_L)+(1-\alpha)(1/\tau_H)\right)+\frac{1}{\left(1-\frac{\log(p_H)-\log(p_L)}{\tau_H}\right)\tau_H}\\ &= -\alpha(1/\tau_L)+\left\{\left(1-\frac{\log(p_H)-\log(p_L)}{\tau_H}\right)^{-1}-(1-\alpha)\right\}(1/\tau_H),\end{aligned}$$

which is negative given $\tau_L \ll \tau_H$. This also suggests that $\Psi^{-i}_{HH}-\Psi^{-i}_{HL}>0$ because

$$\begin{aligned}\Psi^{-i}(p_H,p_H) &= \left(\frac{1}{2}-\frac{\log(p_H)-\log(p_H)}{2\tau}\right)^{-1}\mathbb{E}\left(-\frac{1}{2\tau}\right)-1.\\ &= \mathbb{E}\left[\frac{1}{\tau}\right]\end{aligned}$$

$$\Psi^{-i}(p_H,p_L) = \frac{1-\alpha}{2p_H x^{HL}\tau_H}.$$

D.4 Steady State under Price Stickiness in the Hotelling Address Model

Pure Strategy Under price stickiness, pure strategy is expressed by

$$p^* = 1+\left(\mathbb{E}\left[1/\tau\right]\right)^{-1}\left(1-\frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta}\theta\beta\Gamma^*\right)^{-1}. \tag{73}$$

Importantly, p^* increases by the term of Γ^* under price stickiness, which increases firm profit and decreases an incentive to deviate from this strategy.

Suppose that firm B follows a pure strategy by choosing p^*: that is, firm B sets p^* as long as firm A sets p^*, where p^* is the same as that obtained in equation (62). Then, if firm A sets p^*, the present value of profits is given by $(1 - W/p^*)/2/(1 - \theta\beta)$.

Then, suppose that if firm A deviates from equilibrium pure strategy by choosing price p^d ($p^d > p^*$ to increase its profits), firm B sets the best response price of $p(p^d)$ as long as firm A survives and continues to set the same price when firm B has a chance to reset its price. Also suppose that $p(p^d)$ is close to p^d such that both firms attract both price-sensitive and insensitive consumers. In this case, the present value of profits becomes the max value of

$$\begin{aligned}&(1-\alpha)\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\frac{\log p^d-\log p^*}{2\tau_H}\right)\frac{1}{1-\theta^2\beta}\\&+\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/2/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right)\end{aligned}\tag{74}$$

by choosing the optimal p^d. Thus, the condition for the pure strategy to hold is rewritten as

$$\begin{aligned}&\left(1-\frac{W}{p^*}\right)\frac{1}{2}\frac{1}{1-\theta\beta}\\\geq&(1-\alpha)\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\frac{\log p^d-\log p^*}{2\tau_H}\right)\frac{1}{1-\theta^2\beta}\\&+\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/2/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right).\end{aligned}$$

Here, we should have $p(p^d) > p^*$ owing to a strategic complementarity ($\Gamma^* > 0$). Thus, firm A earns a larger profit after firm B revises its price from p^* to $p(p^d)$. That is, the profit in the first term on the right-hand side is smaller than that in the second term on the right-hand side, and thus, the following condition,

$$\left(1-\frac{W}{p^*}\right)\frac{1}{2}>\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/2/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right),\tag{75}$$

serves as a sufficient condition for the pure strategy to hold given $p(p^d) > p^*$. The right-hand side of the inequality represents the profit that takes account of the effect that its own price reset (to p^d) affects its rival firm's price in the following periods under price stickiness. Note that p^* given by equation 73 is the optimal price that is determined to maximize its present-value profit by taking this effect into account. This is why p^* embeds the term Γ^* and is higher under sticky prices ($\theta > 0$) than under flexible prices ($\theta = 0$). Thus, a deviation of price from p^* likely decreases firm profit, and thus, the above inequality is likely to hold sticky prices ($\theta > 0$) than under flexible prices ($\theta = 0$).

Mixed Strategy Next, we consider the case of mixed strategy.

Proposition 2 *Suppose mixed strategy. The method of undetermined coefficients enables us to* *solve p_H, p_L, Γ, Γ^*, Γ^ε, Λ^n, Λ^{n*}, and $\Lambda^{n\varepsilon}$ from the following equations:*

$$
\begin{aligned}
0 =& \left(\frac{1}{p_H}\right) \cdot \left\{ s(1-\alpha)\left(\frac{1}{2} - \frac{logp_H - logp_L}{2\tau_H}\right) + (1-s)\left(\frac{1}{2}\right)\right\} \\
&+ \left(1 - \frac{1}{p_H}\right) \cdot \left[s(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + (1-s)\mathbb{E}\left(-\frac{1}{2\tau}\right)\right] \\
&+ \left(1 - \frac{1}{p_H}\right)(1-s)\frac{(1-\theta)\theta\beta}{1-\theta^2\beta}\mathbb{E}\left(\frac{\Gamma^*}{2\tau}\right) \\
&+ \theta\beta\left(1 - \frac{1}{p_H}\right) s\Lambda^{n*}\theta(1-\alpha)\left(\frac{1}{2} - \frac{logp_H - logp_L}{2\tau_H}\right) \\
&- \left(1 - \frac{1}{p_H}\right) s\Lambda^{n*}\theta^2\beta\frac{1}{2},
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
0 =& (1-s)\alpha W \\
&+ (1-s)(1-\alpha)W\left(\frac{1}{2} - \frac{logp_L - logp_H}{2\tau_H}\right) \\
&- (1-s)(1-\alpha)(p_L - W)\frac{1}{2\tau_H} \\
&+ sW\frac{1}{2} - s(p_L - W)\mathbb{E}[1/2/\tau],
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
\Pi_H(p_H, p_L, s) \equiv& (1-s)\left(1 - \frac{W}{p_H}\right)\frac{1}{2} + s(1-\alpha)\left(1 - \frac{W}{p_H}\right)\left(\frac{1}{2} - \frac{logp_H - logp_L}{2\tau_H}\right) \\
=&(1-s)\left\{\alpha\left(1 - \frac{W}{p_L}\right) + (1-\alpha)\left(1 - \frac{W}{p_L}\right)\left(\frac{1}{2} - \frac{logp_L - logp_H}{2\tau_H}\right)\right\} \\
&+ s\left(1 - \frac{W}{p_L}\right)\frac{1}{2} \equiv \Pi_L(p_H, p_L, s),
\end{aligned} \tag{78}
$$

$$
0 = A_t + B_t + C_t + D_t + E_t, \tag{79}
$$

$$
\begin{aligned}
&- s\theta\hat{s}_t^n\left(\frac{1}{p_L} - \frac{1}{p_H}\right)\left\{\frac{\alpha}{2} + (1-\alpha)\frac{logp_H - logp_L}{2\tau_H}\right\} \\
&+ \left(\frac{\hat{p}_{t-1}^A - \varepsilon_t}{p_H}\right)\left(\frac{1}{2}(1-s) + s(1-\alpha)\left(\frac{1}{2} - \frac{logp_H - logp_L}{2\tau_H}\right)\right) \\
&+(1-s)\left(1 - \frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A + \theta\hat{p}_{t-1}^B + (1-\theta)p_t^{B*} + (1-\theta)\varepsilon_t}{2\tau}\right) \\
&+s(1-\alpha)\left(1 - \frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A - \varepsilon_t}{2\tau_H}\right) \\
=&(1-s)(1-\alpha)\left(1 - \frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B - \theta\varepsilon_t + (1-\theta)p_t^{B*}}{2\tau_H},
\end{aligned} \tag{80}
$$

where $A_t, B_t, \cdots, E_t$ are defined below.

Proof is below. When firm A has a chance to optimize higher price at t, it sets $\bar{p}_{H,t}^A$ to maximize

$$
\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\left(1 - \frac{W_{t+k}}{\bar{p}_t^A}\right)\left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t+k}^B}{2\tau}\right)M_{t+k}, \tag{81}
$$

given that firm A chooses the higher price rather than the lower price in the periods following price revision. Regarding the probability that firm B chooses the lower price at t, s_t, we need to differentiate two cases: one is when firm B revises the higher price ($s_t^r = s$) and the other is when firm B does not (s_t^n). Then, the above equation is rewritten as

$$\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(\theta s^n_{t+k}+(1-\theta)s)(1-\alpha)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{L,t+k}}{2\tau_H}\right)\right.$$
$$+\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)\theta(1-s^n_{t+k})\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{H,t+k-1}}{2\tau}\right)$$
$$\left.+\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(1-\theta)(1-s)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log\bar{p}^B_{H,t+k}}{2\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.$$

Noting that $p^B_{H,t+k-1}$ equals $\bar{p}^B_{H,t+k-1}$ with the probability of $1-\theta$, $\bar{p}^B_{H,t+k-2}$ with the probability of $\theta(1-\theta),\cdots,\bar{p}^B_{H,t}$ with the probability of $\theta^{k-1}(1-\theta)$, and $p^B_{H,t-1}$ with the probability of θ^k when $k\geq 1$, we have

$$\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(\theta s^n_{t+k}+(1-\theta)s)(1-\alpha)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{L,t+k}}{2\tau_H}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$$
$$+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)\theta(1-s^n_{t+k})\theta^k\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{H,t-1}}{2\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$$
$$+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)\theta(1-s^n_{t+k})\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-1-k'}\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log\bar{p}^B_{H,t+k'}}{2\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$$
$$+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(1-\theta)(1-s)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log\bar{p}^B_{H,t+k}}{2\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}. \tag{82}$$

Furthermore, we should have indifference of profits from choosing the higher price and choosing the lower price when the higher price is not revised in period t, which is

$$\theta(1-s^n_t)\left(1-\frac{M_t}{p^A_{H,t-1}}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\log p^A_{H,t-1}-\log p^B_{H,t-1}}{2\tau}\right)$$
$$+(1-\theta)(1-s)\left(1-\frac{M_t}{p^A_{H,t-1}}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\log p^A_{H,t-1}-\log\bar{p}^B_{H,t}}{2\tau}\right)$$
$$+(\theta s^n_t+(1-\theta)s)(1-\alpha)\left(1-\frac{M_t}{p^A_{H,t-1}}\right)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t-1}-\log(p_L M_t)}{2\tau_H}\right)$$
$$=\theta(1-s^n_t)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\left(\frac{1}{2}-\frac{\log(p_L M_t)-\log p^B_{H,t-1}}{2\tau_H}\right)\right\}$$
$$+(1-\theta)(1-s)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\left(\frac{1}{2}-\frac{\log(p_L M_t)-\log\bar{p}^B_{H,t}}{2\tau_H}\right)\right\}$$
$$+(\theta s^n_t+(1-\theta)s)\left(1-\frac{1}{p_L}\right)\frac{1}{2}. \tag{83}$$

The first-order condition for the optimal $\bar{p}^A_{H,t}$ is given by

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[\left(\theta s^n_{t+k} + (1-\theta)s\right)(1-\alpha) \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{L,t+k}}{2\tau_H}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1 - s^n_{t+k})\theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{H,t-1}}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1 - s^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k'}}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1-\theta)(1-s) \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k}}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \\
&\quad \cdot \left[\left(\theta s^n_{t+k} + (1-\theta)s\right)(1-\alpha) \left(-\frac{1}{2\tau_H \bar{p}^A_{H,t}}\right) + \left\{\theta(1 - s^n_{t+k}) + (1-\theta)(1-s)\right\} \left(-\frac{1}{2\tau \bar{p}^A_{H,t}}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \left[(1 - s^n_{t+k}) \sum_{k'=1}^{k-1} (1-\theta)\theta^{k-k'} \frac{\partial\log\bar{p}^B_{H,t+k'} / \partial\log\bar{p}^A_{H,t}}{2\tau \bar{p}^A_{H,t}}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \left[(1-\theta)(1-s) \frac{\partial\log\bar{p}^B_{H,t+k} / \partial\log\bar{p}^A_{H,t}}{2\tau \bar{p}^A_{H,t}}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \frac{\partial\log s^n_{t+k} / \partial\log\bar{p}^A_{H,t}}{\bar{p}^A_{H,t} / s^n_{t+k}} \\
&\quad \cdot \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{L,t+k}}{2\tau_H}\right) - \theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{H,t-1}}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \frac{\partial\log s^n_{t+k} / \partial\log\bar{p}^A_{H,t}}{\bar{p}^A_{H,t} / s^n_{t+k}} \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k'}}{2\tau}\right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}
$$

Let us define $p^A_{H,t} \equiv p_H M_t e^{\hat{p}^A_t}$,$\bar{p}^A_{H,t} \equiv p_H M_t e^{p^{A*}_t}$, and $s^n_t \equiv s e^{\hat{s}^n_t}$ as well as $\partial\log\bar{p}^B_{H,t+k} / \partial\log\bar{p}^A_{H,t} \equiv$

Γ^* and $\partial \text{log} s^n_{t+k}/\partial \text{log}\bar{p}^A_{H,t} \equiv \Lambda^{n*}$. Then, the log-linearization leads to

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}}\right) \frac{M_{t+k}}{M_t} \left[(s + s\theta \hat{s}^n_{t+k}) (1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}}\right) \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k}) \theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}}\right) \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k}) \sum_{k'=1}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}}\right) \frac{M_{t+k}}{M_t} \left[(1-\theta)(1-s) \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k} - \log(M_{t+k}/M_t)}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}}\right) \left[(s + s\theta \hat{s}^n_{t+k}) (1-\alpha) \left(-\frac{1}{2\tau_H}\right) + \{1 - s - s\theta \hat{s}^n_{t+k}\} \left(-\frac{1}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}}\right) \left[(1 - s - s\hat{s}^n_{t+k}) \theta (1-\theta^k) \frac{\Gamma^*}{2\tau}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}}\right) \left[(1-\theta)(1-s) \frac{\Gamma^*}{2\tau}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}}\right) \Lambda^{n*} (s + s\hat{s}^n_{t+k}) \\
&\cdot \left[\theta (1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H}\right) - \theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{2\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}}\right) \Lambda^{n*} (s + s\hat{s}^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{2\tau}\right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}
$$

In the steady state, it becomes

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{1}{p_H}\right) \cdot \left\{ s(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H}\right) + (1-s)\left(\frac{1}{2}\right)\right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H}\right) \cdot \left[s(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + (1-s)\mathbb{E}\left(-\frac{1}{2\tau}\right)\right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H}\right) (1-s)(1-\theta^k) \mathbb{E}\left(\frac{\Gamma^*}{2\tau}\right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H}\right) s\Lambda^{n*} \left[\theta(1-\alpha)\left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H}\right) - \frac{1}{2}\theta^{k+1}\right] \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H}\right) s\Lambda^{n*} (\theta - \theta^{k+1}) \frac{1}{2},
\end{aligned}
$$

or

$$
\begin{aligned}
0 =& \left(\frac{1}{p_H}\right) \cdot \left\{ s(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H}\right) + (1-s)\left(\frac{1}{2}\right)\right\} \\
&+ \left(1 - \frac{1}{p_H}\right) \cdot \left[s(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + (1-s)\mathbb{E}\left(-\frac{1}{2\tau}\right)\right] \\
&+ \left(1 - \frac{1}{p_H}\right) (1-s) \frac{(1-\theta)\theta\beta}{1-\theta^2\beta} \mathbb{E}\left(\frac{\Gamma^*}{2\tau}\right) \\
&+ \theta\beta \left(1 - \frac{1}{p_H}\right) s\Lambda^{n*} \theta(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{2\tau_H}\right) \\
&- \left(1 - \frac{1}{p_H}\right) s\Lambda^{n*} \theta^2 \beta \frac{1}{2}. \qquad (84)
\end{aligned}
$$

Because of the last three terms (if Γ^* or Λ^{n*} is nonzero), the steady state under nominal rigidity is different from that without nominal rigidity. Firms take account of the dynamic effect of its price on the rival firm's price in the following periods.

Note that the log-linearized deviation of the term $\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$ disappears because of the steady state condition. The log-linearization proceeds as

$$\begin{aligned}
0 = &\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[(s+s\theta\hat{s}^n_{t+k})(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}-\frac{p^{A*}_t-\log(M_{t+k}/M_t)}{2\tau_H}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[(1-s-s\hat{s}^n_{t+k})\theta^{k+1}\left(\frac{1}{2}-\frac{p^{A*}_t-\hat{p}^B_{t-1}+\log(M_t/M_{t-1})}{2\tau}\right)\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[(1-s-s\hat{s}^n_{t+k})\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{p^{A*}_t-p^{B*}_{t+k'}-\log(M_{t+k'}/M_t)}{2\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[(1-\theta)(1-s)\left(\frac{1}{2}-\frac{p^{A*}_t-p^{B*}_{t+k}-\log(M_{t+k}/M_t)}{2\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp^{A*}_t-M_{t+k}/M_t\right)\left[(s+s\theta\hat{s}^n_{t+k})(1-\alpha)\left(-\frac{1}{2\tau_H}\right)+\{1-s-s\theta\hat{s}^n_{t+k}\}\left(-\frac{1}{2\tau}\right)\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp^{A*}_t-M_{t+k}/M_t\right)\left[(1-s-s\hat{s}^n_{t+k})(1-\theta^k)\frac{\Gamma^*}{2\tau}\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp^{A*}_t-M_{t+k}/M_t\right)\left[(1-\theta)(1-s)\frac{\Gamma^*}{2\tau}\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp^{A*}_t-M_{t+k}/M_t\right)\Lambda^{n*}(s+s\hat{s}^n_{t+k})\\
&\cdot\left[\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}-\frac{p^{A*}_t-\log(M_{t+k}/M_t)}{2\tau_H}\right)-\theta^{k+1}\left(\frac{1}{2}-\frac{p^{A*}_t-\hat{p}^B_{t-1}+\log(M_t/M_{t-1})}{2\tau}\right)\right]\\
&-\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp^{A*}_t-M_{t+k}/M_t\right)\Lambda^{n*}(s+s\hat{s}^n_{t+k})\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{p^{A*}_t-p^{B*}_{t+k'}-\log(M_{t+k'}/M_t)}{2\tau}\right),
\end{aligned}$$

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left[s(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) \left[s(1-\alpha) \left(-\frac{1}{2\tau_H} \right) + (1-s) \mathbb{E} \left(-\frac{1}{2\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t)(1-s)(1-\theta^k) \mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) s \Lambda^{n*} \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) - \frac{1}{2} \theta^{k+1} \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) s \Lambda^{n*} \left(-\frac{1}{2}(\theta - \theta^k) \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(s\theta \hat{s}^n_{t+k} \right) (1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t (-s\hat{s}^n_{t+k}) \theta \frac{1}{2} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) \left[\left(s\theta \hat{s}^n_{t+k} \right) (1-\alpha) \left(-\frac{1}{2\tau_H} \right) + \{ -s\theta \hat{s}^n_{t+k} \} \left(-\frac{1}{2\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) (-s\hat{s}^n_{t+k})(1-\theta^k) \frac{\Gamma^*}{2\tau} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) \Lambda^{n*} \left(s\hat{s}^n_{t+k} \right) \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) - \frac{1}{2} \theta^{k+1} \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) \Lambda^{n*} \left(s\hat{s}^n_{t+k} \right) \left(-\frac{1}{2}(\theta - \theta^{k+1}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[s(1-\alpha) \left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s)\theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{2\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-\theta)(1-s) \left(-\frac{p_t^{A*} - p^{B*}_{t+k} - \log(M_{t+k}/M_t)}{2\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) \left[s(1-\alpha) \left(-\frac{1}{2\tau_H} \right) + (1-s) \left(-\frac{1}{2\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) (1-\theta^k) \mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) s\Lambda^{n*} \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) - \frac{1}{2} \theta^{k+1} \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) s\Lambda^{n*} \left(-\frac{1}{2}(\theta - \theta^k) \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) s\Lambda^{n*} \left[\theta(1-\alpha) \left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H} \right) - \theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{2\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1 \right) s\Lambda^{n*} \sum_{k'=1}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{2\tau} \right).
\end{aligned}
$$

Thus, we have

$$0 = A_t + B_t + C_t + D_t + E_t, \tag{85}$$

where

$$\begin{aligned}
A_t \equiv & \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left[s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) + (1-s)\left(\frac{1}{2}\right)\right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(-1\right) \log(M_{t+k}/M_t) \left[s(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + (1-s)\mathbb{E}\left(-\frac{1}{2\tau}\right)\right] \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(-1\right) \log(M_{t+k}/M_t)(1-s)(1-\theta^k)\mathbb{E}\left(\frac{\Gamma^*}{2\tau}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(-1\right) \log(M_{t+k}/M_t) s\Lambda^{n*}\theta(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(-1\right) \log(M_{t+k}/M_t) s\Lambda^{n*}\left(-\frac{1}{2}\theta\right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[s(1-\alpha)\left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H}\right)\right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s)\theta^{k+1}\left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau}\right)\right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*}\right) \left[s(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + (1-s)\left(-\frac{1}{2\tau}\right)\right] \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*}\right)(1-\theta^k)\mathbb{E}\left(\frac{\Gamma^*}{2\tau}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*}\right) s\Lambda^{n*}\theta(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*}\right) s\Lambda^{n*}\left(-\frac{1}{2}\theta\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right) s\Lambda^{n*}\left[\theta(1-\alpha)\left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{2\tau_H}\right) - \theta^{k+1}\left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{2\tau}\right)\right],
\end{aligned}$$

$$\begin{aligned}
B_t \equiv & \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(s\theta \hat{s}_{t+k}^n\right)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t (-s\hat{s}_{t+k}^n)\frac{1}{2}\theta \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right)\left[(1-\alpha)\left(-\frac{1}{2\tau_H}\right) + \frac{1}{2\tau}\right] s\theta \hat{s}_{t+k}^n \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right)(-s\hat{s}_{t+k}^n)\frac{\Gamma^*}{2\tau} \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right)\Lambda^{n*}\left(s\hat{s}_{t+k}^n\right)\theta(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right)\Lambda^{n*}\left(s\hat{s}_{t+k}^n\right)\left(-\frac{1}{2}\right),
\end{aligned}$$

$$C_t \equiv \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H - 1\right)(-s\hat{s}_{t+k}^n)(-\theta^k)\frac{\Gamma^*}{2\tau}$$

$$D_t \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-\theta)(1-s) \left(-\frac{p_t^{A*} - p_{t+k}^{B*} - \log(M_{t+k}/M_t)}{2\tau} \right) \right],$$

and

$$E_t \equiv \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s) \sum_{k'=1}^{k-1} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right]$$
$$+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(p_H - 1)\, s\Lambda^{n*} \sum_{k'=1}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{2\tau} \right) \right].$$

Note that for $k \geq 1$, we have

$$\mathbb{E}_t[\log(M_{t+k}/M_t)] = \sum_{k'=1}^{k} \mathbb{E}_t \varepsilon_{t+k'} = \rho(1-\rho^k)/(1-\rho) \cdot \varepsilon_t,$$

$$M(\varepsilon_t) \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) = \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t.$$

As for A_t, we have

$$\begin{aligned}
A_t &= M(\varepsilon_t) \left[s(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right] \\
&+ (-1)\, M(\varepsilon_t) \left[s(1-\alpha) \left(-\frac{1}{2\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{2\tau} \right) \right] \\
&+ (-1)\, M(\varepsilon_t)(1-s)\mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) \\
&- (-1)\, \frac{\rho}{1-\rho}\theta \left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho} \right) \varepsilon_t (1-s)\mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ (-1)\, M(\varepsilon_t) s\Lambda^{n*}\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) \\
&+ (-1)\, M(\varepsilon_t) s\Lambda^{n*} \left(-\frac{1}{2}\theta \right) \\
&+ \frac{1}{1-\theta\beta} s(1-\alpha) \left(-\frac{p_t^{A*}}{2\tau_H} \right) + M(\varepsilon_t) s(1-\alpha) \frac{1}{2\tau_H} \\
&+ \frac{\theta}{1-\theta^2\beta} (1-s)(p_t^{A*} - \hat{p}_{t-1}^B + \varepsilon_t)\mathbb{E} \left(-\frac{1}{2\tau} \right) \\
&+ \frac{1}{1-\theta\beta} p_H p_t^{A*} \left[s(1-\alpha) \left(-\frac{1}{2\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{2\tau} \right) \right] \\
&+ \frac{\theta\beta}{1-\theta\beta} p_H p_t^{A*} \mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) - \frac{\theta^2\beta}{1-\theta^2\beta} p_H p_t^{A*} \mathbb{E} \left(\frac{\Gamma^*}{2\tau} \right) \\
&+ \frac{\theta\beta}{1-\theta\beta} p_H p_t^{A*} s\Lambda^{n*}\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} \right) \\
&+ \frac{\theta\beta}{1-\theta\beta} p_H p_t^{A*} s\Lambda^{n*} \left(-\frac{1}{2}\theta \right) \\
&+ \frac{\theta\beta}{1-\theta\beta} (p_H - 1)\, s\Lambda^{n*}\theta(1-\alpha) \left(-\frac{p_t^{A*}}{2\tau_H} \right) \\
&+ (p_H - 1)\, s\Lambda^{n*}\theta(1-\alpha) M(\varepsilon_t) \left(\frac{1}{2\tau_H} \right) \\
&+ \frac{\theta^3\beta}{1-\theta^2\beta} (p_H - 1)\, s\Lambda^{n*}(p_t^{A*} - \hat{p}_{t-1}^B + \varepsilon_t)\mathbb{E} \left(\frac{1}{2\tau} \right).
\end{aligned} \tag{86}$$

As for B_t, we have

$$
\begin{aligned}
\theta\beta\mathbb{E}_t B_{t+1} = &\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(s\theta\hat{s}_{t+k}^n\right)(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t(-s\hat{s}_{t+k}^n)\frac{1}{2}\theta\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\left[(1-\alpha)\left(-\frac{1}{2\tau_H}\right)+\frac{1}{2\tau}\right]s\theta\hat{s}_{t+k}^n\\
&+\sum_{k=2}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\left(-s\hat{s}_{t+k}^n\right)\frac{\Gamma^*}{2\tau}\\
&+\sum_{k=2}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\Lambda^{n*}\left(s\hat{s}_{t+k}^n\right)\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+\sum_{k=2}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\Lambda^{n*}\left(s\hat{s}_{t+k}^n\right)\left(-\frac{1}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
B_t = &\ \theta\beta\mathbb{E}_t B_{t+1}\\
&+\left(s\theta\hat{s}_t^n\right)(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+(-s\hat{s}_t^n)\frac{1}{2}\theta\\
&+\left(p_H-1\right)\left[(1-\alpha)\left(-\frac{1}{2\tau_H}\right)+\mathbb{E}\left(\frac{1}{2\tau}\right)\right]s\theta\hat{s}_t^n\\
&+\left(p_H-1\right)\Gamma^*\mathbb{E}\left(\frac{-s\hat{s}_{t+1}^n}{2\tau}\right)\\
&+\theta\beta\left(p_H-1\right)\Lambda^{n*}s\mathbb{E}_t\left(\hat{s}_{t+1}^n\right)\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+\left(p_H-1\right)\Lambda^{n*}s\mathbb{E}_t\left(\hat{s}_{t+1}^n\right)\left(-\frac{1}{2}\right).
\end{aligned}
\tag{87}
$$

Similarly, C_t and D_t are given by

$$
C_t = \theta^2\beta\mathbb{E}_t C_{t+1} + \left(p_H-1\right)\left(s\hat{s}_t^n\right)\theta\Gamma^*\mathbb{E}\left(\frac{1}{2\tau}\right). \tag{88}
$$

$$
\begin{aligned}
D_t \equiv &\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-\theta)(1-s)\left(-\frac{p_t^{A*}-p_{t+k}^{B*}-\log(M_{t+k}/M_t)}{2\tau}\right)\right]\\
= &\ (1-\theta)(1-s)\left[\frac{1}{1-\theta\beta}\left(-p_t^{A*}\right)+M(\varepsilon_t)\right]\mathbb{E}\left(\frac{1}{2\tau}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-\theta)(1-s)\left(\frac{p_{t+k}^{B*}}{2\tau}\right)\right]\\
= &\ (1-\theta)(1-s)\left[\frac{1}{1-\theta\beta}\left(-p_t^{A*}\right)+M(\varepsilon_t)\right]\mathbb{E}\left(\frac{1}{2\tau}\right)+D_t',
\end{aligned}
\tag{89}
$$

where

$$
D_t' = \theta\beta\mathbb{E}_t D_{t+1}' + (1-\theta)(1-s)\mathbb{E}_t\left[p_t^{B*}\right]\mathbb{E}\left(\frac{1}{2\tau}\right). \tag{90}
$$

As for E_t, we have

$$\begin{aligned}
\frac{E_t}{\{(1-s)+(p_H-1)\,s\Lambda^{n*}\}\,\mathbb{E}\left(\frac{1}{2\tau}\right)} &= \sum_{k=1}^{\infty}\theta^k\beta^k\left[\theta(1-\theta^k)\left(-p_t^{A*}\right)\right] \\
&+ \sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
&+ \sum_{k=1}^{\infty}\theta^k\beta^k\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(\frac{\rho(1-\rho^{k'})\cdot\varepsilon_t}{1-\rho}\right)\right] \\
&= \theta\left(-p_t^{A*}\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right) \\
&+ E_t' \\
&+ \theta\sum_{k=1}^{\infty}\theta^k\beta^k\left[1-\theta^k-\frac{\rho^k-\theta^k}{\rho-\theta}(1-\theta)\right]\frac{\rho\varepsilon_t}{1-\rho} \\
&= E_t' \\
&+ \left(-p_t^{A*}\right)\left(\frac{\theta^2\beta}{1-\theta\beta}-\frac{\theta^3\beta}{1-\theta^2\beta}\right) \\
&+ \theta\left[\frac{\theta\beta}{1-\theta\beta}-\frac{1-\theta}{\rho-\theta}\frac{\theta\beta\rho}{1-\theta\beta\rho}+\frac{1-\rho}{\rho-\theta}\frac{\theta^2\beta}{1-\theta^2\beta}\right]\frac{\rho\varepsilon_t}{1-\rho}, \qquad (91)
\end{aligned}$$

where

$$\begin{aligned}
E_t' &= \sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
\mathbb{E}_t\left[E_{t+1}'\right] &= \sum_{k=2}^{\infty}\theta^{k-1}\beta^{k-1}\mathbb{E}_t\left[\sum_{k'=1}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
E_t' &= \theta\beta\mathbb{E}_t\left[E_{t+1}'\right]+\sum_{k=1}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\mathbb{E}_t\left[p_t^{B*}\right] \\
&= \theta\beta\mathbb{E}_t\left[E_{t+1}'\right]+\frac{(1-\theta)\theta^2\beta}{1-\theta^2\beta}\mathbb{E}_t\left[p_t^{B*}\right]. \qquad (92)
\end{aligned}$$

The log-linearization of the profit indifference equation yields

$$\begin{aligned}
&\theta(1-s-s\hat{s}_t^n)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\hat{p}_{t-1}^A-\hat{p}_{t-1}^B}{2\tau}\right) \\
&+(1-\theta)(1-s)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\hat{p}_{t-1}^A-p_t^{B*}-\varepsilon_t}{2\tau}\right) \\
&+s(1+\theta\hat{s}_t^n)(1-\alpha)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{2\tau_H}\right) \\
=&\theta(1-s-s\hat{s}_t^n)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{2\tau_H}+\frac{\hat{p}_{t-1}^B-\varepsilon_t}{2\tau_H}\right)\right\} \\
&+(1-\theta)(1-s)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{2\tau_H}+\frac{p_t^{B*}}{2\tau_H}\right)\right\} \\
&+s(1+\theta\hat{s}_t^n)\left(1-\frac{1}{p_L}\right)\frac{1}{2}.
\end{aligned}$$

$$
\begin{aligned}
&- s\theta\hat{s}_t^n\left(1-\frac{1}{p_H}\right)\frac{1}{2}+(1-s)\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\frac{1}{2}\\
&+(1-s)\left(1-\frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A+\theta\hat{p}_{t-1}^B+(1-\theta)p_t^{B*}+(1-\theta)\varepsilon_t}{2\tau}\right)\\
&+s\theta\hat{s}_t^n(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+s(1-\alpha)\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\\
&+s(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{2\tau_H}\right)\\
&=-s\theta\hat{s}_t^n\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{2\tau_H}\right)\right\}\\
&+(1-s)(1-\alpha)\left(1-\frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B-\theta\varepsilon_t+(1-\theta)p_t^{B*}}{2\tau_H}\\
&+s\theta\hat{s}_t^n\left(1-\frac{1}{p_L}\right)\frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
&s\theta\hat{s}_t^n\left\{-\left(1-\frac{1}{p_H}\right)\frac{1}{2}+(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\right.\\
&\quad+\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{2\tau_H}\right)\\
&\quad\left.-\left(1-\frac{1}{p_L}\right)\frac{1}{2}\right\}\\
&\quad+\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\left(\frac{1}{2}(1-s)+s(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\right)\\
&+(1-s)\left(1-\frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A+\theta\hat{p}_{t-1}^B+(1-\theta)p_t^{B*}+(1-\theta)\varepsilon_t}{2\tau}\right)\\
&+s(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{2\tau_H}\right)\\
&=(1-s)(1-\alpha)\left(1-\frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B-\theta\varepsilon_t+(1-\theta)p_t^{B*}}{2\tau_H}.
\end{aligned}
$$

$$
\begin{aligned}
&-s\theta\hat{s}_t^n\left(\frac{1}{p_L}-\frac{1}{p_H}\right)\left\{\frac{\alpha}{2}+(1-\alpha)\frac{\log p_H-\log p_L}{2\tau_H}\right\}\\
&\quad+\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\left(\frac{1}{2}(1-s)+s(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{2\tau_H}\right)\right)\\
&+(1-s)\left(1-\frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A+\theta\hat{p}_{t-1}^B+(1-\theta)p_t^{B*}+(1-\theta)\varepsilon_t}{2\tau}\right)\\
&+s(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{2\tau_H}\right)\\
&=(1-s)(1-\alpha)\left(1-\frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B-\theta\varepsilon_t+(1-\theta)p_t^{B*}}{2\tau_H}.
\end{aligned}
\tag{93}
$$

Equations (66), (67), (84), (85), and (93) give solutions for p_H, p_L, s, p_t^{A*} (p_t^{B*}), and $\hat{s}_t^n$, where

A_t to E_t are given by equations (86) to (91) and

$$p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t \tag{94}$$

$$p_t^{B*} = \Gamma \hat{p}_{t-1}^B + \Gamma^* \hat{p}_{t-1}^A + \Gamma^\varepsilon \varepsilon_t \tag{95}$$

$$\begin{aligned}
\mathbb{E}_t p_{t+1}^{B*} &= \mathbb{E}_t \left[\Gamma \hat{p}_t^B + \Gamma^* \hat{p}_t^A + \Gamma^\varepsilon \varepsilon_{t+1} \right] \\
&= \theta \Gamma \left(\hat{p}_{t-1}^B - \varepsilon_t \right) + (1-\theta) \Gamma p_t^{B*} + \Gamma^* p_t^{A*} + \rho \Gamma^\varepsilon \varepsilon_t \\
&= \{\theta\Gamma + (1-\theta)\Gamma^2 + \Gamma^{*2}\} \hat{p}_{t-1}^B + (2-\theta)\Gamma\Gamma^* \hat{p}_{t-1}^A + \{-\theta\Gamma + (1-\theta)\Gamma\Gamma^\varepsilon + \Gamma^*\Gamma^\varepsilon + \rho\Gamma^\varepsilon\} \varepsilon_t
\end{aligned} \tag{96}$$

$$\partial \text{log} \bar{p}_{H,t+1}^B / \partial \text{log} \bar{p}_{H,t}^A = \partial p_{t+1}^{B*} / \partial p_t^{A*} = \Gamma^*, \tag{97}$$

and

$$\hat{s}_t^n \equiv \mathbb{E}_t \hat{s}_t^{n,B} = \Lambda^n \hat{p}_{t-1}^B + \Lambda^{n*} \hat{p}_{t-1}^A + \Lambda^{n\varepsilon} \varepsilon_t \tag{98}$$

$$\begin{aligned}
\mathbb{E}_t \hat{s}_{t+1}^n &= \mathbb{E}_t \left[\Lambda^n \hat{p}_t^B + \Lambda^{n*} \hat{p}_t^A + \Lambda^{n\varepsilon} \varepsilon_{t+1} \right] \\
&= \theta \Lambda^n \left(\hat{p}_{t-1}^B - \varepsilon_t \right) + (1-\theta) \Lambda^n p_t^{B*} + \Lambda^{n*} p_t^{A*} + \rho \Lambda^{n\varepsilon} \varepsilon_t \\
&= \{\theta\Lambda^n + (1-\theta)\Lambda^n\Gamma + \Lambda^{n*}\Gamma^*\} \hat{p}_{t-1}^B + \{(1-\theta)\Lambda^n\Gamma^* + \Lambda^{n*}\Gamma\} \hat{p}_{t-1}^A + \{-\theta\Lambda^n + (1-\theta)\Lambda^n\Gamma^\varepsilon + \Lambda^{n*}\Gamma^\varepsilon + \rho\Lambda^{n\varepsilon}\} \varepsilon_t,
\end{aligned} \tag{99}$$

$$\partial \text{log} s_{t+k}^n / \partial \text{log} \bar{p}_{H,t}^A \equiv \Lambda^{n*}. \tag{100}$$

Aggregate Price and Output Aggregate price index is written as

$$\begin{aligned}
\log P_t &= \int_0^1 \log p_t^j dj \\
&= (1-\theta)(1-s) \log \bar{p}_t \cdot x_1 \\
&\quad + ((1-\theta)s + \theta s_t^n) \log p_{L,t} \cdot x_2 \\
&\quad + \theta(1-s_t^n) \log p_{H,t-1} \cdot x_3,
\end{aligned}$$

where

$$x_1 = (1-\theta)(1-s)\frac{1}{2} + ((1-\theta)s + \theta s_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log \bar{p}_t - \log p_{L,t}}{2\tau_H}\right) + \theta(1-s_t^n)\left(\frac{1}{2} - (\log \bar{p}_t - \log p_{H,t-1})\,\mathbb{E}\left(\frac{1}{2\tau}\right)\right)$$

$$\begin{aligned}
x_2 = &(1-\theta)(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_{L,t} - \log \bar{p}_t}{2\tau_H}\right)\right) \\
&+ ((1-\theta)s + \theta s_t^n)\frac{1}{2} + \theta(1-s_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_{L,t} - \log p_{H,t-1}}{2\tau_H}\right)\right)
\end{aligned}$$

$$x_3 = (1-\theta)(1-s)\left(\frac{1}{2} - (\log p_{H,t-1} - \log \bar{p}_t)\,\mathbb{E}\left(\frac{1}{2\tau}\right)\right) + ((1-\theta)s + \theta s_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_{H,t-1} - \log p_{L,t}}{2\tau_H}\right) + \theta(1-s_t^n)\frac{1}{2}.$$

Thus, the log-linearized aggregate price divided by M_t, which equals minus log-linearized aggregate output, is given by

$$\begin{aligned}
-\text{log} Y_t = \log P_t - \log M_t &= (1-\theta)(1-s)(\log p_H + p_t^*) \cdot x_1 \\
&\quad + s(1 + \theta \hat{s}_t^n)(\log p_L) \cdot x_2 \\
&\quad + \theta(1 - s - s\hat{s}_t^n)(\log p_H - \varepsilon_t + \hat{p}_{t-1}) \cdot x_3,
\end{aligned}$$

where

$$
\begin{aligned}
x_1 &= (1-\theta)(1-s)\frac{1}{2} + s\left(1+\theta\hat{s}_t^n\right)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} - \frac{p_t^*}{\tau_H}\right) \\
&+ \theta(1-s-s\hat{s}_t^n)\left(\frac{1}{2} - (p_t^* - \hat{p}_{t-1} + \varepsilon_t)\,\mathbb{E}\left(\frac{1}{2\tau}\right)\right) \\
&= (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
&+ \theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2} - s(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta(1-s)\left(-p_t^* + \hat{p}_{t-1} - \varepsilon_t\right)\mathbb{E}\left(\frac{1}{2\tau}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= (1-\theta)(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H} + \frac{p_t^*}{2\tau_H}\right)\right) + s\left(1+\theta\hat{s}_t^n\right)\frac{1}{2} \\
&+ \theta(1-s-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H} + \frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right)\right) \\
&= (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right) + s\frac{1}{2} \\
&+ (1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta s\hat{s}_t^n\frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right) + \theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= (1-\theta)(1-s)\left(\frac{1}{2} - (\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{2\tau}\right)\right) + s\left(1+\theta\hat{s}_t^n\right)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H} - \frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right) + \theta(1-s-s\hat{s}_t^n)\frac{1}{2} \\
&= (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) \\
&- (1-\theta)(1-s)\left(\hat{p}_{t-1} - p_t^* - \varepsilon_t\right)\mathbb{E}\left(\frac{1}{2\tau}\right) + \theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) + s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\log P_t - \log M_t = (1-s)\left(\log p_H\right) &\cdot \left\{(1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ s\left(\log p_L\right) \cdot \left\{(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right) + s\frac{1}{2}\right\} \\
&+ \theta(1-s)\left(-\varepsilon_t\right) \cdot \left\{(1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ (1-\theta)(1-s)p_t^* \cdot \left\{(1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \theta s\hat{s}_t^n\left(\log p_L\right) \cdot \left\{(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right) + s\frac{1}{2}\right\} \\
&- \theta s\hat{s}_t^n\left(\log p_H\right) \cdot \left\{(1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \theta(1-s)\hat{p}_{t-1} \cdot \left\{(1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ (1-\theta)(1-s)\left(\log p_H\right) \\
&\cdot \left\{\theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2} - s(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta(1-s)\left(-p_t^* + \hat{p}_{t-1} - \varepsilon_t\right)\mathbb{E}\left(\frac{1}{2\tau}\right)\right\} \\
&+ s\left(\log p_L\right)\left\{(1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta s\hat{s}_t^n\frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right)\right\} \\
&+ s\left(\log p_L\right)\left\{\theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right)\right\} \\
&+ \theta(1-s)\left(\log p_H\right)\left\{-(1-\theta)(1-s)\left(\hat{p}_{t-1} - p_t^* - \varepsilon_t\right)\mathbb{E}\left(\frac{1}{2\tau}\right) + \theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \theta(1-s)\left(\log p_H\right)\left\{s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\log P_t - \log M_t = \hat{P}_t = -\hat{Y}_t \\
=& \theta(1-s)(-\varepsilon_t) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \{(1-\theta)(1-s)p_t^* - \theta s \hat{s}_t^n (\log p_H) + \theta(1-s)\hat{p}_{t-1}\} \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \theta s \hat{s}_t^n (\log p_L) \cdot \left\{ (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right) + s\frac{1}{2}\right\} \\
&+ (1-\theta)(1-s)(\log p_H) \\
&\cdot \left\{ \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2} - s(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta(1-s)(-p_t^* + \hat{p}_{t-1} - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{2\tau}\right)\right\} \\
&+ s(\log p_L)\left\{ (1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{2\tau_H} + \theta s \hat{s}_t^n \frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{2\tau_H}\right)\right)\right\} \\
&+ s(\log p_L)\left\{ \theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right)\right\} \\
&+ \theta(1-s)(\log p_H)\left\{ -(1-\theta)(1-s)(\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{2\tau}\right) + \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{2\tau_H}\right)\right\} \\
&+ \theta(1-s)(\log p_H)\left\{ s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{2\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2}\right\}, \qquad (101)
\end{aligned}
$$

where

$$
\hat{p}_t = \theta(\hat{p}_{t-1} - \varepsilon_t) + (1-\theta)p_t^*. \qquad (102)
$$